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CONTENTS

Arisz, W. H.: "Absorption and transport by the tentacles of Drosera capensis." I. Active transport of asparagine in the parenchyma cells of the tentacles, p. 2.

Burgers, J. M.: "On the influence of the concentration of a suspension upon the sedimentation velocity (in particular for a suspension of spherical particles)", p. 9.

Weitzenböck, R., und W. J. Bos: "Zur projektiven Differentialgeometrie der Regelflächen im R_4". (Achte Mitteilung), p. 17.

Scholte, J. G.: "On the Stoneley-wave equation." I. (Communicated by Prof. J. D. van der Waals), p. 20.

Kulk, W. van der: "Zur Theorie der verallgemeinerten Pfaff'schen Gleichungen." (Communicated by Prof. J. A. Schouten), p. 26.

Veen, S. C. van: "Stark konvergente Entwicklungen für die vollständigen elliptischen Integrale erster und zweiter Art." IV. (Communicated by Prof. J. G. van der Corput), p. 32.

Veen, S. C. van: "Ueber die Entwicklung der unvollständigen elliptischen Integrale erster und zweiter Art in stark konvergenten Reihen." IV. (Communicated by Prof. J. G. van der Corput), p. 37.

Monna, A. F.: "Sur quelques inégalités de la théorie des fonctions et leurs généralisations spatiales." I. (Communicated by Prof. W. van der Woude), p. 43.

Bungenberg de Jong, H. G., and B. Kok: "Tissues of prismatic cells containing Biocolloids." IV. Morphological changes of the complex coacervate gelatine + gum arabic in consequence of a pH change of the medium flowing along the membrane. (Communicated by Prof. H. R. Kruyt), p. 51.

Bungenberg de Jong, H. G., and E. G. Hoskam: "Effect of neutral salts on the composition of complex coacervate (gelatine + gum arabic) and equilibrium liquid at constant pH and constant mixing proportion of the two colloids in the total system. (Communicated by Prof. H. R. Kruyt), p. 59.

Bungenberg de Jong, H. G., and B. Kok: "Tissues of prismatic cells containing Biocolloids." V. Morphological changes of the complex coacervate gelatine-gum arabic owing to the addition of salts resp. non-electrolytes to the liquid flowing past the membrane. (With two plates.) (Communicated by Prof. J. van der Hoeve), p. 67.

Bungenberg de Jong, H. G.: "Tissues of prismatic cells containing Biocolloids." VI. Location of coëxisting coacervates and equilibrium liquid in the cells. Morphological model of the plant cell. (Communicated by Prof. J. van der Hoeve), p. 76.

Hermes, J. J., and D. R. de Vletter: "Contribution to the petrography of Bintan (Riouw-Lingga Archipelago). (With one plate and one map.) (Communicated by Prof. L. Rutten), p. 82.

Raadshooven, B. v., and J. Swart: "On rocks from Karimon (Riouw Archipelago). (With one map.) (Communicated by Prof. L. Rutten), p. 89.

Hoogenraad, H. R., and A. A. de Groot: "New observations on the feeding of Vampyrella lateritia (Fres.) Leidy." (Communicated by Prof. J. Boeke), p. 97.

Révész, G.: "Das Problem des Ursprungs der Sprache." I. (Communicated by Prof. A. P. H. A. de Kleyn), p. 105.

Botany. — *Absorption and transport by the tentacles of Drosera capensis. I. Active transport of asparagine in the parenchyma cells of the tentacles.* By W. H. ARISZ.

(Communicated at the meeting of December 27, 1941.)

§ 1. *Introduction.*

OUDMAN (1936) showed that the tentacles of Drosera capensis are organs which can take up asparagine and caffeine with their glands and transport these substances by means of the pedicels to the leaf. He pointed out some differences in the absorption of asparagine and caffeine, which he attributed to the different transport-tracks for these substances. With caffeine we mainly observe a diffusion through the vacuole, with asparagine the transport would take place through the protoplasm, the protoplasm being much more permeable to caffeine than to asparagine. ARISZ and OUDMAN (1937) pointed out the different behaviour of these substances with regard to the causing of aggregation in the parenchyma cells of the pedicels. They connected the better transport of asparagine with the aggregating action of this substance, whereas caffeine does not cause aggregation but granulation.

From a research of ARISZ and OUDMAN (1938) on the absorption and the transport of asparagine through the leaves of Vallisneria it had appeared that asparagine-absorption is an active accumulation-process, which is dependent on a normal supply of oxygen. There is similarity between this process and the absorption of salts (HOAGLAND, STEWARD). This has given rise to investigating whether the absorption and the transport of asparagine by the tentacles of Drosera may be an analogous process. In this communication it will be shown that the transport of asparagine is carried out by the living parenchyma cells of the pedicels, and is dependent on the supply of oxygen; besides it is an accumulationprocess.

In making the analyses and experiments I have met with excellent help from Dr. J. OUDMAN, Miss J. VAN WEERDEN and Miss J. VAN DER SCHANS, for which I tender them my best thanks. In connection with the above the experiments have been divided into three series: series O in 1938, series W in 1938—1940 and series S in 1941.

§ 2. *Method.*

The experiments here discussed were made in the months of April to October 1938—1941 with plants of Drosera capensis grown from seeds. The plants were of various ages. Young plants have a higher content of N-compounds than old ones and absorb more asparagine too. Therefore we used material of the same age for an experiment as far as this was possible. The length of the leaves varies from 2.5 to 5 cms. The method used to examine the transport in the tentacles was already described before (ARISZ and OUDMAN 1937). The extremities of the marginal-tentacles on either side of the leaves are put between strips of agar. In the 2 % agar solution the substance which is to be taken up, is also present. If necessary other substances can be dissolved in the agar as well. The lamina the marginal tentacles of which are lying in the agar on both sides of the leaf, is for the rest entirely free from the agar-strips, so that transport to the leaf can only take place through the tentacles. The number of tentacles placed between the agar strips, amounts to about 150 with short leaves, with longer ones to 320 and more. Just as in previous experiments the variability in the behaviour of the leaves of various ages and of various plants was more or less eliminated by numbering the leaves of six plants from one to six and subsequently combining into one series one leaf of each plant, inserted on a different level. In this way 6 series are obtained each containing 6 leaves. These show a fair correspondence in nitrogen-content. (See on this subject OUDMAN 1936, Table VI, p. 386). The substances of which the transport has been investigated are mainly the following: asparagine, glycocoll, alanine, leucine, urea, methyl-urea, thiourea, urotropine, caffeine, KH_2PO_4, K_2HPO_4, NH_4Cl and KNO_3. In this communication asparagine will be discussed in particular. The quantity taken up has been ascertained according to the micro-Kjeldahl method. The quantity of nitrogen is related to the fresh weight of the leaves.

All experiments were made in a room of a constant temperature at 25° C. in the dark.

§ 3. *Transport by the parenchyma cells of the pedicel.*

The tentacles of Drosera possess an ellipsoidical gland, in which there are a number of gland-cells. (HOMÈS 1929). The function of these gland-cells is evidently in the first place secreting the viscid fluid which always covers the tentacle-glands. On stimulation of the tentacles or on stimulating the leaf a strong and sometimes prolonged secretion takes place (DARWIN 1875). The gland is likewise capable of taking up a great number of substances. Where the pedicel is covered with a cuticle, water and dissolved substances cannot penetrate it. This becomes evident when from a tentacle both gland and pedicel are brought into a caffeine-solution. The caffeine that has penetrated immediately precipitates in the vacuole. From this very sensitive reaction it appears that caffeine only penetrates into the gland and this substance cannot diffuse through the outer wall of the pedicel. Only into those places of the pedicel, where there are so-called papillae, the caffeine enters. After the substances have penetrated through the gland, they must be carried through the pedicel to the leaf. The length of the pedicels of the marginal-tentacles amounts to 3 to 4 mms. In the pedicel one or two spiral vessels occur. When the caffeine ist taken up, we see from the occurrence of the precipitate in the vacuole, how the caffeine passes on from cell to cell and starts in every cell from the apical end. From this KOK rightly concluded that caffeine is transported through the vacuole and not through the cell-wall or the protoplasm. In her experiments on tentacles without glands, transport through the spiral vessel would have been possible, because on removing the tentacle-gland the spiral vessels were opened. The course of the transport through the parenchyma-cells, however, proves, that hardly any transport of caffeine takes place through the spiral vessel.

On the analogy of this ARISZ and OUDMAN have also assumed for the transport of asparagine that this would take place in the parenchyma cells and not in the spiral vessel.

Now the question arises whether it will be possible to prove that the parenchyma cells of the pedicel actually transport asparagine. The transport might be brought about in a different way. For instance the tentacle-gland might be an organ that takes up substances from outside and then gives them up again to the liquid in the spiral vessel. The transport of the dissolved substance might then take place either by means of active secretion of the tentacle gland inwards or by means of suction, resulting from a suction pressure which may arise in the leaf for instance through transpiration. Though in the transport experiments of OUDMAN (1936) and ARISZ and OUDMAN (1937) this possibility was taken into account and the leaves were put in closed glass boxes, the air of which was saturated with water vapour, yet it seemed desirable to obtain more data on this matter. Therefore two series of experiments have been made: one in which the transpiration by the leaf was increased by putting the leaves in dry air, and a second series in which sugar was added to the agar, so that water was osmotically absorbed from the tentacles. This arrangement caused in the tentacles a flow of water in exactly the opposite direction to that in which the transport of the absorbed substances towards the leaf had to take place. The result of the experiments leaves no doubt but the transport of asparagine is entirely independent of currents in the tentacles which are caused by the transpiration of the leaf or by absorption of water from the tentacle by osmotically acting substances in the agar.

Three series of experiments have been taken: Series A, experiments in which the leaves were in a dry atmosphere; series B experiments in which through addition of sucrose to the agar water absorption from the tentacles took place and series C experiments in which the leaf was exposed to greater transpiration and sugar was also added to the agar.

Series A. Increase of the transpiration of the lamina. The dry air was obtained by bringing a saturated solution of Na_2SO_4, $(NH_4)_2SO_4$ or Na_2CO_3 into the closed glass boxes in which the Drosera leaves were. The relative humidity of the air above these liquids is 93 % (20° C.), 81.1 % (25° C.), 92 % (18.5° C.) respectively. From the experiments mentioned in table I it may be concluded that with intact tentacles the uptake and the transport of asparagine and caffeine is little or not affected by an increased transpiration of the lamina. In an atmosphere saturated with water vapour an equally strong absorption takes place as in an atmosphere of 89—90 % humidity. In these experiments two difficulties present themselves. The first difficulty shows when the quantity of N taken up is related to the fresh weight of the leaf. If the series of leaves are weighed before the beginning of the experiment, the mucilage of the tentacle-glands would be included, unless it is first removed from the glands by washing. The quantity of secreted fluid is not slight in proportion to the fresh weight of the leaf. So 2 series of 6 leaves appeared to weigh 489 and 522 mg., secretion included; after washing and removal of the secretion 317 and 326 mg. resp.; so that the secretion amounted to 172 and 196 mg.; that is more than 50 % of the fresh weight of the leaves. As washing of the secretion before the experiment would injure the tentacles too much, it is necessary to determine the fresh weight at the end of the absorption period. If the absorption takes place in an environment in which during the experiment there is a loss of water from the leaves, relating the nitrogen content on the fresh weight could give rise to an error. The absorbed quantity of N expressed in fresh weight per thousand will be found too high in this case. In these cases it is preferable to compare the absolute values of N taken up per series. As each series possesses an equal number of leaves, this method is useful in such cases, though it is not particularly accurate. If we take this into account, it is evident that an increase of absorption owing to transpiration of the leaf is out of the question, either with the absorption of caffeine, or with that of asparagine.

TABLE I. Influence of increased transpiration of the leaves on the uptake by the tentacles. In experiments 4 and 5 sugar 0.35 mol. is added to the agar.

	Absorption	Nitrogen increase in γ		Nitrogen increase in $^0/_{00}$ fresh weight	
		humid air	dry air	humid air	dry air
1	24 hours 1/20 mol. asparagine	254	286	0.90	0.97
2	24 hours 1/20 mol. asparagine	332	220	0.75	0.70
3	24 hours 1/20 mol. asparagine	358	352	1.05	1.04
4	24 hours 1/20 mol. asparagine	241	210	0.84	0.81
5	24 hours 1/20 mol. asparagine	604	638	1.77	1.75

In these experiments a second difficulty arises, i.e. that the tentacles under the influence of asparagine curve from the agar, especially in a humid environment. Owing to this the absorption of the asparagine from the agar is not so great. To meet this difficulty a series of experiments C has been made, in which transpiration of the leaves also took place, but inflection of the tentacles was prevented.

Series B. In order to get a flow of water in the tentacles in the direction of the leaf to the tentacle-gland sugar was dissolved in the agar. Owing to the water absorption from the tentacles and from the leaf, the increase in N-content will become too large, if it is calculated as the difference in N-content at the beginning of the experiment, related on the fresh weight at the beginning of the experiment, and the N-content at the end of the experiment related on the final fresh weight. Therefore the absolute values N, in γ per series have also been given in Table II. From the data obtained it appears, that

TABLE II. Influence of adding sugar to the agar with the asparagine on the uptake by the tentacles.

Absorption	Sugar conc. in agar	Nitrogen increase in γ		Nitrogen increase in $^0/_{00}$ fresh weight	
24 hours 1/20 mol. asparagine	none	306		0.70	
	0.2 mol	274		0.84	
	0.3 mol	372		1.32	
	0.4 mol	398		1.35	
24 hours 1/20 mol. asparagine	none	200		0.74	
	0.3 mol	212		0.87	
	0.45 mol	310		1.26	
	0.6 mol	312		1.34	
		1	2	1	2
24 hours 1/20 mol. asparagine (two experiments)	none	202		0.76	
	0.45 mol	220	496	1.17	1.46
	0.60 mol	280	514	1.37	1.84
	0.75 mol	198	406	1.11	1.83
	0.90 mol	176	236	1.12	1.51

sugar-concentrations higher than 0.3 mol. prevent the tentacles from curving out off the agar. These get less turgescent on account of the loss of water and cannot make a curvature. So at the same time this provides a method of preventing the inflection of tentacles. As far as it was possible to ascertain this by experiments, the strength of the absorption and the transport of asparagine does not alter in the least through a sugar-concentration of 0.3—0.4 mol. On the contrary, because the tentacles remain in closer contact with the agar, the absorption is much increased. In the highest sugar-concentrations, however, the absorption does decrease. A plasmolysis of the parenchyma cells of the pedicel does not show in these circumstances even in high sugar-concentrations. Nor is this to be expected, as the sugar-solution cannot pass through the impermeable outer wall. It is rather a shrivelling of the whole pedicel that takes place, which is shown in the slighter turgor of the tentacles, owing to which inflections cannot arise and especially the parts close below the gland clearly show folds of the wall. In spite of the fact that a flow of water is brought about in the tentacles towards the gland, yet absorption and transport takes place towards the leaf.

Series C. In these experiments (table I exp. 4 and 5) there is a sugar-concentration of 0.35 mol. in the agar, while the lamina lies either in humid air above water or in dry air above saturated $(NH_4)_2SO_4$. A curving-out of the tentacles from the agar cannot take place under the circumstances, while through the transpiration of the leaf a tension is developed by suction, which carries water towards the leaf through the tentacle. Under these circumstances there was not found any influence of the water-flow through the tentacle on the transport of asparagine and caffeine either.

§ 4. *Influence of withdrawal of oxygen on the uptake of asparagine.*

Seeing that the absorption of asparagine by the leaves of Vallisneria spiralis is a process that takes place in the presence of oxygen (ARISZ and OUDMAN 1938) it seemed desirable to ascertain whether the uptake of asparagine by the Drosera-tentacles is also a process dependent on the presence of oxygen. From a number of experiments, comprised in table III it appears that this is actually the case. Without an exception the absorption is slighter when oxygen is absent. The withdrawal of oxygen was obtained by conducting into a McIntosh and Fildes anaerobic jar first purified N-gas and next combining the oxygen still present to hydrogen-gas by means of a palladium-catalysator. In one of the experiments the oxygen was probably not completely removed, in the remaining the transport was considerably checked and amounts to only 0 to 14 % of the transport in aerobic conditions.

In entire correspondence with this OUDMAN (1936) found, that aethernarcosis checks the uptake of asparagine. From table III experiment 6 it appears that also under the influence of 1/300 mol. KCN, which was added to the agar-strips, the absorption is greatly inhibited.

These experiments prove satisfactorily that absorption and transport of asparagine are sensitive to decrease of the oxygen-pressure and inhibited by KCN and aether-narcosis.

Now it seemed important to find out, whether tentacles of which the glands had been removed, were still capable of taking up asparagine and if even then this process would be sensitive to the withdrawal of oxygen. In this way it must be possible to prove that transport of asparagine through the pedicels is an active process, for which the presence of oxygen in the living cells is required.

OUDMAN has already ascertained whether tentacles from which the glands have been cut-off, still take up asparagine. For tentacles without glands he found an uptake of 73 % of those with glands.

In table III (exp. 7, 8 and 9) a summary has been given of some experiments. From the figures it appears that also in tentacles without glands the transport is stronger in the presence of oxygen. It is true the percentage that has been taken up anaerobically is higher, but that is not to be wondered at. In the first place these tentacles are in an abnormal condition owing to the injury caused on cutting off the glands and besides the spiral vessel has been opened, so that a free diffusion and flow may take place in it. As also from agar to which 0.35 mol. sucrose has been added, more is taken up aerobically than anaerobically, it is evident that the transport takes place in the parenchyma cells of the pedicels.

TABLE III. Influence of oxygen withdrawal and KCN on the uptake of asparagine. In exp. 7, 8 and 9 the glands have been cut off before the beginning of the experiment.

	Absorption	Nitrogen increase in $^0/_{00}$ fresh weight		Anaerobic as $^0/_0$ of normal
		normal	anaerobic	
1	24 hours 1/10 mol.	1.45	0.21	14.5$^0/_0$
2	24 hours 1/10 mol.	1.71	0.44	25.7$^0/_0$
3	24 hours 1/20 mol.	0.83	0.—	0 $^0/_0$
4	48 hours 1/20 mol.	1.60	0.06	4 $^0/_0$
5	24 hours 1/20 mol. with 0.35 mol sugar in agar	1.15	0.15	12.2$^0/_0$
			KCN $\frac{1}{300}$	
6	24 hours 1/20 mol.	1.14	0.26	22.8$^0/_0$
	tentacles without glands		anaerobic	
7	24 hours 1/20 mol.	0.79	0.41	
8	24 hours 1/20 mol.	0.51	0.10	
9	24 hours 1/20 mol. with 0.35 mol sugar in agar	0.41	0.20	

§ 5. *Absorption of asparagine is an accumulation-process.*

Now that it has been shown that the transport of asparagine must take place by the living tentacle-cells and is dependent on aerobic respiration, it is important to know whether this process like the taking up of asparagine by the leaf cells of Vallisneria is an accumulation process, in which the substance is accumulated against a concentration gradient. OUDMAN has shown by experiments in which the tentacles were cut off before the analysis, that the absorbed asparagine does not remain in the tentacles but also arrives in the lamina of the leaf. Where the asparagine taken up is found in the leaf, has not further been ascertained. It has, however, appeared from OUDMAN's research, that in the leaf there occurs no formation of protein from asparagine. In how much other conversions of asparagine take place is unknown. For the present we will presume that for the first 24 hours the asparagine in the leaf continues unaltered and is equally distributed over all leaf-cells. Though this supposition will not be quite correct, as the asparagine concentration will probably be higher nearer to the marginal tentacles, yet it enables us to get an impression of how the concentration of asparagine would be, if it were present in solution in the cell-sap of the leaf cells. On the question whether the asparagine taken up arrives in the cell-sap completely or partly, we have no data. On the ground of what is known about the absorption by other tissues, however, it is likely that a considerable part of the asparagine is present in solution in the vacuole.

Table IV gives an impression of the strength of the accumulation. In weaker concentrations the accumulation-factor is greater and amounts in the case of 1/1280 mol. and 1/5120 mol. asparagine to about 37. With this it is satisfactorily shown that the transport of asparagine may take place against a concentration gradient of this substance.

TABLE IV. Accumulation of asparagine.

Mol. asparagine conc. in agar strips	Nitrogen in $^0/_{00}$ of the fres weight	Mol. asparagine conc. in leaves	Accumulation factor
0.05	1.47	0.0525	1
0.0125	1.90	0.0679	5
0.003125	1.50	0.0536	17
0.000781	0.80	0.0286	37
0.000195	0.20	0.0071	37

§ 6. *Summary and Discussion.*

From the preceding it appears that the tentacles of Drosera capensis are capable of transporting asparagine through the pedicels. As already shown by OUDMAN this process has a high temperature-coefficient. It has been shown that this transport is dependent on a proper oxygen-supply to the living tentacle-cells. By narcosis and by KCN it is inhibited. Accumulation occurs, so that transport can take place against a concentration gradient. The tentacle-gland has no specific function in the absorption. Tentacles, of which the glands have been removed, are also capable of taking up asparagine actively. Transport through the spiral vessels of the pedicel does not figure in it. Neither by sucking water from the leaf towards the gland, by bringing it into agar to which osmotically working sucrose has been added, nor by increasing the transpiration of the leaf and sucking water towards the leaf through the tentacles, absorption or transport can be noticeably accelerated or retarded.

Accordingly it has been proved that the transport of asparagine in the tentacles is a transport-process in parenchyma cells, which is not dependent on a concentration gradient of the substance transported and therefore cannot be a diffusion process. MASON and MASKELL's theory about activated diffusion does not obtain for this process. It is, however, closely connected with what is known about the absorption and the transport of salts in the cortex of the root. We shall go further into the nature of the process in a following communication in connection with results obtained about the transport of other substances.

The data obtained on the strength of the transport enable us to calculate the strength of the transport in the pedicels. When 6 leaves wih 1200 tentacles take up 300 γ nitrogen in 24 hours, $\frac{1}{4}$ γ N. is transported per tentacle, i.e. $\frac{33}{28}$ γ asparagine. The diameter of a tentacle just below the gland amounts to about 0.04 mm. So $\frac{33}{28}$ γ asparagine is transported through a surface of 0.00126 mm^2 in 24 hours, i.e. 0.039 mg. asparagine per mm^2 per hour. If the transport takes place through the protoplasm, this figure rises considerably. Reliable data on the transport in parenchyma cells in root, stalk or leaf are not known to me. So we come to comparing the transport in the tentacles with that in the sieve tubes. For transport in the stalk (MÜNCH) 10.7—63.3 mg. per mm^2 per hour was found, for transport of assimilates from a beanleaf (BIRCH–HIRSCHFELD) 5 mg. per mm^2 per hour, for supply of assimilates to fruit (MÜNCH) 4.7—6 mg. per mm^2 per hour, for the rate of transport of sugars in the stalk in cotton (MASON and MASKELL) 2.3 mg per mm^2 per hour. From this it appears that the rate of transport in the sieve-tubes is more than 100 times faster than the transport in the parenchyma cells of the tentacles. Therefore it appears on comparison with the transport in the sieve-tubes that the rate of the latter is much greater. For the present there is no reason to assume that the transport in the sieve-tube would be a process that is analogous with the active transport in the Drosera tentacles.

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Hydrodynamics. — *On the influence of the concentration of a suspension upon the sedimentation velocity* (in particular for a suspension of spherical particles) *). By J. M. BURGERS. (Mededeeling N⁰. 42 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft).

(Communicated at the meeting of December 27, 1941.)

15. With the aid of the results obtained in sections **11.**—**14.** we will now attempt to calculate the influence which a given particle experiences from all the surrounding particles, in a field extending indefinitely in all directions and everywhere possessing the same average number of particles per unit volume. It will be seen in **17.** that a difficulty still remains in the problem, in so far as there occurs an integral, the value of which depends upon the way the integration is carried out. By prescribing a certain definite way a particular value is obtained, which to the author would appear the one best adapted for the present purpose, but the problem cannot yet be considered as being wholly settled.

We begin with the summation of the velocities induced in a particle A in consequence of the presence of the other particles ("particles B"). These particles can be taken together in groups, each group being situated at some definite distance r_i from A; the number of particles per group being n_i. The contribution by each group will be calculated upon the assumption that we may use the mean value of (54) over a surface $r = constant$. Restoring the factor $F/8\pi\eta$ the total amount becomes:

$$\delta u_{\mathrm{I}} = \Sigma u_i = -\frac{5Fa^3}{24\pi\eta} \sum \frac{n_i}{r_i^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

In working out the sum it is not necessary to proceed far: from a certain distance r_m onward it is sufficiently accurate to make use of the integral:

$$n \int_{r_m}^{\infty} 4\pi r^2\,dr\,(1/r^4) = 4\pi n/r_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where n is the average number of particles per unit of volume. The distance r_m is defined by:

$$n\,.\,(4\pi r_m^3/3) = 1 + \Sigma n_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

the summation extending just as far as we take separate terms in (55).

For purposes of comparison we write:

$$\delta u_{\mathrm{I}} = -\lambda_{\mathrm{I}}\, n\, s\, u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58a)$$

where $s = 4\pi a^3/3$, $u_0 = F/6\pi\eta a$ (compare 30c), and:

$$\lambda_{\mathrm{I}} = \frac{15a}{4r_m} + \frac{15a}{16\pi n} \sum \frac{n_i}{r_i^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58b)$$

16. The evaluation of (58*b*) is possible only when we possess a statistical theory of the distribution of the particles in the neighbourhood of a given one. As it is not a part of our task to develop such a theory here, we shall restrict to the consideration of a few typical cases.

We might assume in the first place that the surrounding particles may take all positions

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*) Continued from these Proceedings **44**, 1941, p. 1184.

relatively to A with equal probability, provided they do not penetrate into A. The minimum distance of the centre of a particle B from the centre of A then will be: $2a$; hence we apply eq. (58*b*) with $r_m = 2a$, leaving out the sum. This gives:

$$\lambda_{\mathrm{I}} = 15/8 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (59a)$$

A second assumption is that owing to the action of repulsive forces there may be a minimum value βa for r_m, exceeding $2a$, while otherwise there shall be no restriction, nor any preference for the possible positions of B. In that case we find:

$$\lambda_{\mathrm{I}} = 15/(4\beta). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (59b)$$

When the number of particles per unit volume becomes large, the repulsive forces between them may enforce a type of distribution in which the average distances between neighbouring particles become approximately equal. For purposes of calculation the arrangement may be compared with certain types of regular arrangements. We may assume, *e.g.*, that the average values of r_i and n_i for the first few groups of particles surrounding A approximately are the same as those which are found in a simple cubical lattice with spacing l. In that case we have: $n l^3 = 1$, while the first few groups are determined by:

$$\begin{array}{llll} r_1 = l; & r_2 = l\sqrt{2}; & r_3 = l\sqrt{3}; & r_4 = 2l \\ n_1 = 6; & n_2 = 12; & n_3 = 8; & n_4 = 6. \end{array}$$

Equation (57) then gives: $r_m = 1{,}990\ l$, and from (58*b*) we obtain: $\lambda_{\mathrm{I}} = 4{,}96\ a/l =$ $= 4{,}96\ a\, n^{1/3}$. Instead of the simple cubical arrangement we also might consider the face-centred lattice as a possible picture for the average arrangement of the particles. If the spacing has the value l, we have: $n l^3 = 4$; and the first few groups are determined by:

$$\begin{array}{llll} r_1 = \frac{1}{2} l\sqrt{2}; & r_2 = l; & r_3 = l\sqrt{3/2}; & r_4 = l\sqrt{2} \\ n_1 = 12; & n_2 = 6; & n_3 = 24; & n_4 = 12. \end{array}$$

In this case eq. (57) gives: $r_m = 1{,}486\ l$; from (58*b*) we obtain: $\lambda_{\mathrm{I}} = 7{,}57\ a/l = 4{,}77\ a\, n^{1/3}$. As the latter value does not differ greatly from the one found with the simple cubical arrangement, we can write:

$$\lambda_{\mathrm{I}} \simeq 4{,}9\, a\, n^{1/3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (59c)$$

as an approximate expression to be applied in the case of a more or less regular average spatial distribution of the particles.

As will be evident from sections **2.**, **11.** and **12.**, and from footnote 11), the resultant effect of the "induced velocities" corresponds to what usually is described as the result of an apparent increase of the viscosity. When the retardation δu_{I} had been calculated by means of EINSTEIN's formula for the specific increase of the viscosity, the result would have been:

$$\delta u_{\mathrm{I}} = \frac{F}{6\pi\eta a}\left(\frac{1}{1 + 2{,}5\, ns} - 1\right) \simeq -2{,}5\, ns\, u_0,$$

so that in that case we should find: $\lambda_{\mathrm{I}} = 2{,}5$. It will be seen that the values of λ_{I} given by eqs. (59*a*)—(59*c*) all are smaller. This is due to the circumstance that EINSTEIN's formula refers to the motion in bulk of the suspension, and applies to those cases where the field of flow does not change greatly over distances of the order of the mean distance between two particles. In the case treated here it is otherwise: the velocity gradients have their greatest values immediately around a given particle, *i.e.* in a part of space where a second particle cannot penetrate as easily as elsewhere. Hence in the immediate neighbourhood of a given particle the "effective viscosity" remains below the

EINSTEIN value, and the difference becomes more marked when repulsive forces lead to an increase of the extent of the "empty region".

17. We now come to the summation of the velocities described by (37*a*). For this purpose we consider a definite particle *B*, and take in view the velocities communicated to it by all other particles, each of which in turn must be considered as a particle *A*. In a similar way as was done in **16.** the particles *A* surrounding a given particle *B* can be taken together in groups, each group lying at a definite distance r_i from *B*. The contributions derived from all particles can be united into the expression:

$$\delta u_{\mathrm{II}} = \Sigma\, n_i\, (\bar{u})_i + n \int_{r_m}^{\infty} 4\pi r^2\, dr\, \bar{u}\,. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

where $\bar{u}$ is the mean value of u_m over a surface $r = constant$; $(\bar{u})_i$ in particular being the mean value for the surface $r = r_i$; while r_m is defined by (57).

The expression (60) can be written:

$$\delta u_{\mathrm{II}} = \left\{ \Sigma\, n_i\, (\bar{u})_i - n \int_a^{r_m} 4\pi r^2\, dr\, \bar{u} \right\} + n \int_a^{\infty} 4\pi r^2\, dr\, \bar{u} \quad . \quad . \quad (60a)$$

The last integral on the right hand side will be replaced by: $n \iiint dx\, dy\, dz\, u_m$, the integration being extended over the whole space outside of the spherical surface $r = a$. Making use of (37*a*) this integral can be decomposed into:

$$n \iiint dx\, dy\, dz\, u + \tfrac{1}{6}\, n\, a^2 \iiint dx\, dy\, dz\, \triangle\, u.$$

The first integral of this sum is not a convergent expression: its value depends upon the way in which the integration is carried out. This apparently means that the problem of the unlimited field is not determinate in itself, and a particular condition must be supplied to specify the value of this integral. We adopt the value given in eq. (34), which was obtained by performing the integration with respect to *dy* and *dz* first, the integration with respect to *dx* coming afterwards, as the value given in (34) is the same as the one which is obtained for the case of a field enclosed by impenetrable walls (compare equation (19), section 6.). In this way the first term of the sum becomes: $-n\, s\, u_0$[14]. — The second integral of the sum can be reduced to a surface integral, extended over the spherical surface $r = a$, the integrand being: $-\partial u/\partial r$, the value of which can be derived from (30*b*). This second integral reduces to: $+\tfrac{1}{2}\, n\, s\, u_0$. Combining the results we obtain:

$$n \int_a^{\infty} 4\pi r^2\, dr\, \bar{u} = -\tfrac{1}{2}\, n\, s\, u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60b)$$

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[14] When the mean value u over a surface $r = constant$ is calculated immediately from (29) one obtains: $(F\varkappa/12\pi\eta)\, e^{-\sigma}\, (1 + 2/\sigma)$. Substitution of this expression into the integral $n \int_a^{\infty} 4\pi r^2\, dr\, u$ leads to the result: $4\, n\, F/3\, \varkappa^2 \eta$.

When in calculating $n \iiint dx\, dy\, dz\, u$ the integration first is performed over a surface $y = constant$, the integration with respect to dy coming last, the result becomes: $2nF/\varkappa^2\eta$.

These results cannot have any sense, as both of them depend upon the parameter $\varkappa$, which to a certain extent is arbitrary.

In this way we have got around the difficulties encountered in section **7**. It is possible, however, that the result is not final, and the circumstance that in any actual case the field is bounded by the walls of the vessel containing the suspension as yet may play a deciding part.

The value of $\bar{u}$ to be used in the expression between the { } in (60*a*) can be derived from eq. (37*b*) of section **14**. After restoration of the factor $F/8\pi\eta$ it becomes: $\bar{u} = u_0 \, a/r$. Making use of this result and of (60*b*) it is found that (60*a*) can be written:

$$\delta u_{\mathrm{II}} = -\lambda_{\mathrm{II}} \, n \, s \, u_0 \, . \qquad (61a)$$

where:

$$\lambda_{\mathrm{II}} = \frac{3}{2}\frac{r_m^2}{a^2} - 1 - \frac{3}{4\pi a^2 n}\sum \frac{n_i}{r_i} \qquad (61b)$$

This expression will be worked out for the same cases as have been considered in connection with the calculation of λ_{I}. When the surrounding particles may take all positions relatively to B with equal probability, from a distance $r_m = 2a$ onward, we can discard the sum and find:[15]

$$\lambda_{\mathrm{II}} = 5. \qquad (62a)$$

In the case where the minimum distance is βa (with $\beta > 2$):

$$\lambda_{\mathrm{II}} = \tfrac{3}{2}\beta^2 - 1 \qquad (62b)$$

For a distribution of the distances of the nearest neighbours corresponding to that found in a simple cubical lattice: $\lambda_{\mathrm{II}} = 0{,}663 \, l^2/a^2 - 1 = 0{,}66 \, a^{-2} n^{-2/3} - 1$; and for a distribution corresponding to that found in a face-centred cubical lattice: $\lambda_{\mathrm{II}} = 0{,}267 \, l^2/a^2 - 1 = 0{,}67 \, a^{-2} n^{-2/3} - 1$; giving as an average value:

$$\lambda_{\mathrm{II}} \cong 0{,}67 \, a^{-2} n^{-2/3} - 1 \qquad (62c)$$

18. The quantity designated by δu_{II} represents the velocity which would be acquired by a particle B, of density equal to that of the liquid, in consequence of the fields of flow which are produced by the sedimenting particles A surrounding it. When the original density is restored to B, this particle moreover will acquire the velocity $u_0 = F/6\pi\eta a$ under the action of its own weight (compare eq. 30c); at the same time it will also experience the resultant effect of the "induced velocities", indicated by δu_{I}. Hence the resulting velocity of the particle will become:

$$u_{\mathrm{res}} = F/6\pi\eta a + \delta u_{\mathrm{I}} + \delta u_{\mathrm{II}} \qquad (63)$$

Although terms of the second order, such as may be called forth by the combination of the effects considered, have not been taken into account, it is probable that the accuracy of the result expressed by (63) will be increased, when in the expressions (58*a*) and (61*a*), for δu_{I} and δu_{II} respectively, we replace u_0 by the resulting velocity u_{res}. Indeed, the effects denoted by δu_{I} and δu_{II} refer to fields of flow set up by sedimenting

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[15] The same value is obtained from eq. (24) of the first part of this paper (these Proceedings **44**, 1941, p. 1051), when k_2 and k_3 are replaced by zero. With the values given in **7**. we then have:

$$(8\pi\eta/F)\,\delta u_{\mathrm{II}} = (N/\Omega)\iiint_G dx\,dy\,dz\,\Phi_m = -n \, . \, 80\,\pi a^2/9,$$

where N is the total number of particles contained in the vessel, so that $n = N/\Omega$. Introducing s and u_0 we obtain: $\delta u_{\mathrm{II}} = -5\,n\,s\,u_0$.

particles, and thus are proportional to the actual velocity acquired by a particle. Making this substitution, we obtain:

$$u_{\text{res}} = F/6\pi\eta a - (\lambda_{\text{I}} + \lambda_{\text{II}})\, ns\, u_{\text{res}} \quad . \quad . \quad . \quad . \quad . \quad (63a)$$

from which:

$$u_{\text{res}} = \frac{F}{6\pi\eta a} \frac{1}{1 + (\lambda_{\text{I}} + \lambda_{\text{II}})\, ns} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where, according to (35): $F = s(\varrho_p - \varrho)g$.

19. Now that we have obtained a provisional expression for the value of the sedimentation velocity in an infinitely extending field, it would be necessary to return to the case of a suspension enclosed in a vessel. However, we will first give attention to the motion of a *cloud of particles of finite extent*, in a field which itself is unlimited. The influence exerted by the particles upon each other's motion in this case will increase the velocity of fall, which may acquire values greatly exceeding the sedimentation velocity of a single particle. It is possible — and it actually occurs in many cases — that the velocity acquired by the whole mass becomes of such magnitude, that it is no more allowed to leave out the inertia terms from the equations of motion. Nevertheless we shall provisionally assume that the linear equations, in which the inertia terms have been neglected, can be applied (cases can be constructed in which no serious error is to be expected); afterwards some attention will be given to the possibilities for a more general treatment.

When we keep to the linear equations of motion, the resulting velocity of any particle in principle can be found by adding together (*a*) the velocity it derives from the force acting upon the particle itself; (*b*) the velocities induced in consequence of the presence of the surrounding particles; and (*c*) the velocities which it will derive from the fields of flow set up by the surrounding particles. This third contribution is given by:

$$\delta u_{\text{II}} = \Sigma\, u_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

where the sum extends over all particles of the cloud, with the exception of the particle *B* for which the velocity must be found. The value of this sum depends upon the dimensions and the form of the cloud; upon the distribution of the particles through the cloud; and upon the position of the particle *B* within the cloud. We assume that the number of particles per unit volume (n) has the same value everywhere in the cloud, and that the form of the cloud is spherical, with radius R_0. When the particle *B* is situated not too near to the surface of the cloud (in the following lines we will limit ourselves to the consideration of such particles), the expression (65) can be written:

$$\delta u_{\text{II}} = \Sigma n_i \,.\, (\bar{u})_i + n \iiint\limits_{r > r_m} dx\, dy\, dz\, u_m \,. \quad . \quad . \quad . \quad . \quad (66)$$

where the integral extends over the space outside of a spherical surface with radius r_m, again defined by (57). On account of (37*a*) we have:

$$\iiint\limits_{r > r_m} dx\, dy\, dz\, u_m = \iiint\limits_{r > r_m} dx\, dy\, dz\, u + \tfrac{1}{6}\, a^2 \iiint\limits_{r > r_m} dx\, dy\, dz \triangle u.$$

By means of a direct calculation it is found that the second integral on the right hand side has the value zero in the case considered. Consequently it is possible to transform (66) into:

$$\delta u_{\text{II}} = \{ \Sigma\, n_i \,.\, (\bar{u})_i - n \int_0^{r_m} 4\pi r^2\, dr\, \bar{u} \} + n \iiint dx\, dy\, dz\, u \quad . \quad (67)$$

The triple integral here is extended over the whole cloud; in the integral occurring between the { } it is necessary therefore to take $r=0$ as the lower limit (instead of $r=a$, as was done in (60a) above). As $\bar{u}=u_0 a/r$, this latter integral remains convergent for $r=0$, and has the value: $2\pi a r_m^2 u_0$; hence the quantity between the { } in (67) can be written:

$$-\lambda^* n s u_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68a)$$

where:

$$\lambda^* = \lambda_{\mathrm{II}} + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68b)$$

20. In calculating the value of:

$$U = n \iiint dx\, dy\, dz\, u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

it is to be observed that in the present case, where the number of particles is finite, difficulties concerning the convergence will not occur. Hence it is not necessary to make use of the solution applied in the case of an infinitely extending assemblage of particles, which was given in **10.**, and we can base our calculations immediately upon the formulae developed by STOKES.

The most convenient way is to make use of the expression for u, given in **9.**, *viz.*: $u = u_{\mathrm{I}} + u_{\mathrm{II}} = \triangle\Psi - \partial^2\Psi/\partial x^2 + \partial^2\varphi/\partial x^2$. We first calculate the integrals of the functions Ψ and φ; the velocity U afterwards can be derived by means of differentiations. Instead of working with the function Ψ given in (26) we now can use the much simpler expression: [16])

$$\Psi_{\mathrm{Stokes}} = F r / 8\pi\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

It must be observed that a construction of the type as was proposed in **10.** can be applied also to the present case; we come back to this point in **22.**

An elementary calculation gives:

$$\iiint dx\, dy\, dz\, \frac{Fr}{8\pi\eta} = \frac{F}{8\eta}\left(R_0^4 + \tfrac{2}{3} R^2 R_0^2 - \tfrac{1}{15} R^4\right) \quad . \quad . \quad (71a)$$

$$-\iiint dx\, dy\, dz\, \frac{Fa^2}{24\pi\eta r} = \frac{-Fa^2}{12\eta}\left(R_0^2 - \tfrac{1}{3} R^2\right) \quad . \quad . \quad . \quad . \quad . \quad (71b)$$

where R is the distance of the particle B from the centre of the spherical cloud (R_0 being the radius of the cloud itself). The necessary differentiations can be performed when we write: $R^2 = x^2 + y^2 + z^2$, the origin of the system of coordinates being taken at the centre of the cloud. We then find:

$$U = \frac{nF}{15\eta}\left(5R_0^2 - x^2 - 2y^2 - 2z^2\right) + \frac{nFa^2}{18\eta} \quad . \quad . \quad . \quad . \quad (72a)$$

and in a similar way for the components in the directions of the other axes:

$$V = \frac{nF}{15\eta} xy; \qquad W = \frac{nF}{15\eta} xz \quad . \quad . \quad . \quad . \quad . \quad (72b, c)$$

21. The quantities U, V, W are of an order of magnitude quite different from that of the quantities which thus far have played a part in our calculations. Discarding all terms of less importance we can say that the motion of the particles of the cloud to a first approximation is described by the equations (72a)—(72c), in which, moreover, the last term of (72*a*) safely can be neglected. This motion can be decomposed into a *general motion of the whole cloud* with the constant velocity:

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[16]) Compare eq. (30a).

$$u_{\text{cloud}} = \frac{4\,nF}{15\,\eta} R_0^2 = \frac{4}{3}\pi R_0^3 \,.\, nF \,.\, \frac{1}{5\pi\eta R_0} \quad \ldots\ldots \quad (73)$$

and an *interior motion* with the components:

$$\left.\begin{aligned} u_{\text{interior}} &= \frac{nF}{15\,\eta}(R_0^2 - R^2 - y^2 - z^2) \\ v_{\text{interior}} &= \frac{nF}{15\,\eta}\,xy\,; \quad w_{\text{interior}} = \frac{nF}{15\,\eta}\,xz \end{aligned}\right\} \quad \ldots\ldots \quad (74)$$

These latter quantities satisfy the equation of continuity. At the surface of the cloud:

$$x\,u_{\text{interior}} + y\,v_{\text{interior}} + z\,w_{\text{interior}} = 0 \quad \ldots\ldots \quad (75)$$

from which it appears that the interior motion is tangential to this surface. Hence the spherical form of the cloud and the constant value of the number of particles per unit volume are retained during the motion.

It will be evident that the quantities given by (72*a*)—(72*c*) do not only represent the velocities of the particles in the cloud, but also *that of the liquid itself*. The liquid in the interior of the cloud thus is carried along by the particles it contains.

The motion described by eqs. (72*a*)—(72*c*) is the same as which is found for a *liquid sphere* of radius R_0, acted upon by a continuously distributed force of effective magnitude nF per unit volume, and falling in another liquid, provided both liquids possess the same viscosity [17]. Actually we must expect that owing to the presence of the particles in the sphere, the latter will possess an effective viscosity greater than that of the surrounding liquid. That this is not apparent from the equations developed must be ascribed to the circumstance that in calculating δu_{II} by means of (65) we simply have summed the amounts u_m, without considering the influence of all the other particles upon each term of this sum. Now that the sum has assumed a magnitude much larger than all other velocities, this influence certainly can no longer be neglected.

22. The results arrived at make it appear more promising to start from a different point of view, related to that of section **10**. The system of forces acting upon the liquid and the cloud of particles can be analysed into the following components:

a) a force ϱg per unit volume, acting throughout the whole field, and balanced by a pressure gradient $\partial p/\partial x = \varrho g$ (assumed to be present also in the particles);

b) a continuous force of magnitude nF per unit volume, assumed to act throughout the volume of the cloud of particles;

c) a set of "equilibrium systems" of the type considered in **9.**, each system having its centre at the centre of a particle.

In order to reduce as far as possible the difficulties which may arise at the boundaries of the cloud, it is necessary to choose the parameter $\varkappa$, which occurs in the formulae describing the equilibrium systems, in such a way that $1/\varkappa$, while still being large in comparison with the average distance between neighbouring particles, at the same time is small compared with the dimensions of the cloud.

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[17] Compare H. LAMB, Hydrodynamics (6th Ed., Cambridge 1932), Art. **337**, 2⁰ (p. 600). The resistance experienced by a liquid sphere, moving with the velocity U_s in another liquid, is given by: $6\pi\eta a U_s (2\eta + 3\eta')/(3\eta + 3\eta')$, η' being the viscosity of the liquid of the sphere. When $\eta' = \eta$, this formula reduces to: $5\pi\eta a U_s$.

The "effective force" nF mentioned in the text is the total force acting per unit volume of the cloud, diminished by ϱg, as follows from: $nF = ng(\varrho_P - \varrho)s$.

We will not work out the calculation of the field of motion according to the scheme indicated, and restrict to the following observations:

The field of force considered under *b*) will produce a motion of the cloud as a whole, which motion will be the same as that of a mass of liquid with density $\varrho + nF/g = \varrho + ns(\varrho_p - \varrho)$, moving amidst a liquid of density ϱ. It is reasonable to assume that the liquid represented by the cloud will possess the effective viscosity $\eta' = \eta\ (1 + 2{,}5\,n\,s)$. In many cases which are encountered in actual circumstances, the motion of this mass of liquid will be such that inertia effects, both in its interior and in the surrounding liquid, cannot be neglected. A theoretical calculation then may become impossible, and experimental investigation often must be called to assistance.

Superposed upon the motion of the cloud as a whole, there will be the motion of the particles relatively to the liquid under the action of the force systems, mentioned under *c*). When the particles are sufficiently small, the sedimentation velocity usually will be extremely small in comparison with that of the cloud as a whole. The relative motion then can be calculated upon lines, similar to those followed in **15**.—**18**. There may be found some difference in the value of λ_{II}, connected with the fact that the cloud is of finite extent; also the corrections for particles near to the boundary of the cloud will be different.

Examples of the motion of such clouds of particles, carrying along with themselves the liquid contained in the cloud, are often found in nature. We mention the motion of the fog; that of clouds heavily loaded with dust particles (beautiful demonstration experiments can be made with cold smoke); the phenomena presented by certain clouds which sometimes emerge from volcanic lavas and are loaded so heavily with ashes or scoriae, that they flow down the slopes of the mountain with very great velocities[18]); water currents loaded with silt such as have been considered in DALY's theory of the formation of submarine canyons and are illustrated by beautiful experiments made by KUENEN[19]). Attention also should be called to the phenomenon known as eviction[20]).

In many of these cases the particles will be so heavy that STOKES' law of resistance no longer can be applied to them, and a different law (ultimately a quadratic law) of resistance should be used. Moreover, in the motion of such clouds and currents turbulence usually plays a large part; apart form the influence it has upon the motion of the mass as a whole, it is of importance as it brings about an intense mixing and diffusion, which counteracts the sedimentation of the particles and thus keeps them much longer suspended. In all these cases a decomposition of the system of forces into three parts in the way as indicated above, and the consideration of the general motion of the suspension as that of a liquid of increased density and viscosity, will afford a valuable help in analysing the phenomena presented.

It must be remarked that when it is necessary to consider the frictional forces due to the turbulent motion, attention should be given also to the influence of the suspended particles upon the magnitude of these forces.

In the last part of this paper we hope to come back to the problem of the sedimentation in a suspension enclosed in a vessel.

(To be continued.)

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[18]) The explanation of the "nuées ardentes" as the flow of turbulent clouds of ashes down the slopes of the mountain in consequence of the force of gravity has been given by G. L. L. KEMMERLING; compare *e.g.* his paper: "De controverse uitgeschoten gloedwolken (nuées ardentes d'explosion dirigées) of lawinen gloedwolken (nuées ardentes d'avalanche)", De Ingenieur **47**, 1932, p. A 129.

[19]) Compare: PH. H. KUENEN, Experiments in connection with DALY's hypothesis on the formation of submarine canyons, Leidsche Geologische Mededeelingen **8**, 1937, p. 327; Density currents in connection with the problem of submarine canyons, Geological Magazine **75**, 1938, p. 241.

[20]) Compare: N. SHAW, The air and its ways (Cambridge 1923), p. 103.

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Achte Mitteilung). Von R. WEITZENBÖCK und W. J. BOS.

(Communicated at the meeting of December 27, 1941.)

Wir behandeln in dieser Mitteilung einige Sätze über die Flächen F_2^3 des R_4, die durch drei gegebene Geraden allgemeiner Lage gehen.

§ 24.

Es seien a^2, α^2 und p^2 drei Geraden allgemeiner Lage. Ihre Transversale L schneidet sie in den drei Punkten

$$\left.\begin{aligned} &P_1 = (\alpha^2 p^2 a)(au') = 0, \quad P_2 = (p^2 a^2 \alpha)(\alpha u') = 0 \\ \text{und} \qquad &(a^2 \alpha^2 p)(pu') = -P_1 - P_2 = 0 \end{aligned}\right\} \quad . \quad . \quad (221)$$

Es seien P_3, P_4 und P_5 drei weitere Punkte mit $(P_1 P_2 P_3 P_4 P_5) \neq 0$, P_3 auf a^2, P_4 auf α^2 und P_5 auf der dritten Geraden p^2 gelegen. Auf jeder Regelfläche F_2^3, von der a^2, σ^2 und p^2 Erzeugende sind, liegt ein durch P_3 und P_4 gehender Kegelschnitt K, der p^2 in einem Punkte $P_1 + P_2 + \alpha P_5$ trifft. Die Punkte von K sind dann durch die drei Erzeugenden a^2, α^2 und p^2 projektiv auf die der Leitlinie $P_1 P_2 = L$ bezogen.

Als Parameterdarstellung für K erhalten wir, wenn $t = 0$ dem Punkte P_3, $t = \infty$ dem Punkte P_4 und $t = 1$ dem Punkte $P_1 + P_2 + \alpha P_5$ entspricht:

$$\varrho P_K = \beta \,.\, P_3 + t\,[-\beta \,.\, P_3 - \gamma \,.\, P_4 + P_1 + P_2 + \alpha \,.\, P_5] + t^2 \gamma \,.\, P_4 \quad (222)$$

Für die Punkte von L setzen wir

$$\varrho P_L = P_1 + t \,.\, P_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (223)$$

sodass der allgemeine Flächenpunkt x auf F_2^3 gegeben ist durch

$$x = P_L + \lambda \,.\, P_K;$$

d.h. wir haben

$$\left.\begin{aligned} \varrho x = P_1 \,.\, (1 + \lambda t) + P_2 \,.\, (t + \lambda t) + P_3 \,.\, (\lambda \beta - \lambda t \beta) + &\\ P_4 \,.\, (-\lambda t \gamma + \lambda t^2 \gamma) + P_5 \,.\, \lambda t \alpha & \end{aligned}\right\} (224)$$

Nehmen wir also das Simplex der fünf Punkte P_i als Koordinatensimplex, so sind die Punkte $X(t, \lambda)$ der allgemeinsten Fläche F_2^3 durch die drei Geraden a^2, α^2 und p^2 dargestellt durch

$$\left.\begin{aligned} \sigma X_1 &= 1 + \lambda t \\ \sigma X_2 &= t + \lambda t \\ \sigma X_3 &= \lambda \beta (1 - t) \\ \sigma X_4 &= \lambda t \gamma (t - 1) \\ \sigma X_5 &= \lambda t \alpha \end{aligned} \qquad \begin{array}{l} \text{Die drei Erzeugenden } a^2, \alpha^2, p^2 \text{ gehören} \\ \text{zu den Werten } t = 0, \infty, 1;\ \gamma = 0 \text{ gibt} \\ \text{die Leitlinie } L. \end{array}\right\} \quad . \quad (225)$$

Es gibt also ∞^3 Flächen F_2^3 durch die drei gegebenen Geraden, entsprechend den drei Parametern α, β, γ.

Aus den letzten drei der Gleichungen (225) erhält man

$$U = \alpha X_3 X_4 + \gamma X_3 X_5 + \beta X_5 X_4 = 0 \quad . \quad . \quad . \quad . \quad (226)$$

Weiters ergibt sich aus eben denselben Gleichungen durch Elimination von t, λ und γ:

$$V = \beta (X_1 - X_2) X_5 - X_3 (\alpha X_2 - X_5) = 0 \quad . \quad . \quad . \quad (227)$$

$U = 0$ stellt einen zweifach-ausgearteten quadratischen R_3 dar, dessen singuläre Gerade die Leitlinie L ist. $V = 0$ gibt einen Hyperkegel mit P_4 als Spitze. $U = 0$ und $V = 0$ schneiden sich in einer Fläche vierter Ordnung, die in die Ebene $X_3 = X_5 = 0$ und in die Fläche $F_2{}^3$ zerfällt.

§ 25.

Wenn die durch die Gleichungen (225) dargestellte $F_2{}^3$ durch einen gegebenen Punkt y mit den Koordinaten Y_i bezgl. $P_1P_2P_3P_4P_5$ gehen soll, so müssen in erster Linie nach (226) und (227) die Gleichungen gelten

$$\left.\begin{aligned} U(Y, Y) &= \alpha Y_3 Y_4 + \gamma Y_3 Y_5 + \beta Y_4 Y_5 = 0 \\ V(Y, Y) &= \beta (Y_1 - Y_2) Y_5 - \alpha Y_2 Y_3 + Y_3 Y_5 = 0. \end{aligned}\right\} \quad . \quad (228)$$

Hieraus finden wir z.B. bei $Y_5 \, (Y_1 - Y_2) \neq 0$:

$$\beta = \frac{Y_3 (\alpha Y_2 - Y_5)}{Y_5 (Y_1 - Y_2)}, \quad \gamma = -\frac{Y_4 (\alpha Y_1 - Y_5)}{Y_5 (Y_1 - Y_2)} \quad . \quad . \quad . \quad . \quad (229)$$

Dies kann man in (225) einsetzen und findet so *die Parameterdarstellung der* ∞^1 *Flächen* $F_2{}^3$*, die durch drei gegebene Geraden und einen Punkt* Y *gehen.* Die durch Y gehende Erzeugende entspricht dem Werte

$$t_\alpha = \frac{\alpha Y_2 - Y_5}{\alpha Y_1 - Y_5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

Schreibt man also t_α, also α vor, so ist $F_2{}^3$ gegeben, d.h. man hat dann diejenige Fläche $F_2{}^3$, die durch vier Geraden mit gemeinsamer Transversale L bestimmt ist, wobei die vierte Gerade durch Y und den auf L liegenden Punkt $P_1 + t_\alpha P_2$ gegeben ist.

Aus (226) bis (228) lassen sich α, β und γ eliminieren. Man erhält

$$\begin{vmatrix} X_3 X_4 & X_4 X_5 & X_3 X_5 & 0 \\ -X_2 X_3 & (X_1 - X_2) X_5 & 0 & X_3 X_5 \\ Y_3 Y_4 & Y_4 Y_5 & Y_3 Y_5 & 0 \\ -Y_2 Y_3 & (Y_1 - Y_2) Y_5 & 0 & Y_3 Y_5 \end{vmatrix} = X_3 X_5 Y_3 Y_5 \, . \, Q(X, Y) = 0,$$

wobei

$$\left.\begin{aligned} Q(X, Y) = -X_3 X_4 Y_5 (Y_1 - Y_2) - X_4 X_5 Y_2 Y_3 + X_3 X_5 Y_1 Y_4 - \\ -(X_1 - X_2) X_5 Y_3 Y_4 - X_2 X_3 Y_4 Y_5 + X_1 X_4 Y_3 Y_5. \end{aligned}\right\} \quad (231)$$

$Q = 0$ ist die Gleichung eines quadratischen Kegels mit der Spitze Y. *Alle Flächen* $F_2{}^3$ *durch* Y *füllen somit diesen Kegel aus.*

Ebenso schliesst man: $Q(Y, Z) = 0$ *ist die Bedingung dafür, dass es eine* $F_2{}^3$ *durch die drei Geraden* a^2, α^2 *und* p^2 *gibt, die beide Punkte* Y *und* Z *enthält.*

Es ist naheliegend, diese Ergebnisse mit den projektiven Komitanten J der drei Geraden in Zusammenhang zu bringen. Man findet alle J, wenn man die Kovarianten der drei

Geraden mit einer Reihe x und die Invarianten der vier Geraden a^2, α^2, p^2 und $(xy)_{ik}$ ermittelt. Die letzteren sind alle, wie ich früher bewiesen habe 1) von der Gestalt:

$$J_{a,\alpha p,\pi\varrho} = (a\,\alpha^2\,p^2)\,(a\,\pi^2\,\varrho^2) = 1_{23,45} \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

Ist hier $4_{ik} = (xy)_{ik}$ nur einmal vorhanden, so haben wir z.B.

$$J_{4,12,23} = \Sigma\,(xy)_{ik}\,(a^2\,\alpha^2)_i\,(\beta^2\,p^2)_k = (xa^2\,\alpha^2)\,(y\beta^2\,p^2) - (ya^2\,\alpha^2)\,(x\beta^2\,p^2),$$

führt also auf Kovarianten der Gestalt

$$X_{12} = (xa^2\,\alpha^2), \quad Y_{12} = (ya^2\,\alpha^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

Ist dagegen in (232) $4_{ik} = (xy)_{ik}$ zweimal anwesend, so ergibt sich die Komitante

$$Q_1 = J_{1,24,34} = (a\alpha^2\,xy)\,(ap^2\,xy). \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

Es gilt dann

$$Q_1 = J_{1,24,34} = -J_{2,14,34} = -J_{3,24,14}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (235)$$

sodass sich nur eine solche Komitante ergibt.

Bezüglich der Kovarianten mit nur einer Reihe x ergibt sich leicht, dass sie alle von der Gestalt (233) sind.

Schreiben wir jetzt (231) in der Gestalt

$$Q = (X\,Y)_{45}\,(X\,Y)_{13} + (X\,Y)_{45}\,(X\,Y)_{32} + (X\,Y)_{25}\,(X\,Y)_{43}, \quad . \quad (236)$$

so ergibt eine leichte Rechnung $Q = 2Q_1$, sodass also auch

$$Q_1 = (a\alpha^2\,xy)\,(ap^2\,xy) = 0$$

den Kegel (231) darstellt.

Für $(xy)_{ik} = \pi_{ik}$ ist

$$(a\alpha^2\,\pi^2)\,(ap^2\,\pi^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (237)$$

ein quadratischer Linienkomplex, nämlich der Ort jener Geraden, durch die sich Ebenen legen lassen, die a^2, α^2 und p^2 schneiden. Einem Punkte y entspricht bezüglich dieses Komplexes der Kegel $Q(X, Y)$. Liegen zwei Punkte y und z auf einer Komplexgeraden, so gibt es eine Fläche F_2^3 durch a^2, α^2 und p^2, die diese beiden Punkte enthält.

Sind die drei Geraden a^2, α^2 und p^2 drei aufeinanderfolgende Erzeugende einer allgemeinen Regelfläche im R_4, so ist der quadratische Linienkomplex (237) durch die Differentialkomitante

$$(01^2\,\pi^2)\,(02^2\,\pi^2) = 0_{1\pi,2\pi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (238)$$

gegeben.

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1) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 245—252 (1939).

Geophysics. — *On the* STONELEY-*wave equation.* I. By J. G. SCHOLTE. (Communicated by Prof. J. D. v. d. WAALS.)

(Communicated at the meeting of November 29, 1941.)

§ 1. *Introduction.*

As early as 1899 an investigation was made by KNOTT[1]) about the relations between the amplitudes of plane waves, vibrating in the plane of incidence, which are reflected and refracted at a plane surface of separation of two infinite elastic solids.

In this problem we have always to do with 5 waves, namely the incident wave, the reflected longitudinal and transversal waves and the two refracted waves. The 4 boundary conditions (continuity of the normal and tangential components of motion and those of tension) are expressed by 4 equations, which are linear with respect to the amplitudes and the coefficients of which depend on the material constants and the angle of incidence i. Therefore the 4 amplitudes of the refracted and the reflected waves (one longitudinal and one transversal) can generally be expressed in the amplitude of the incident wave.

A particular wave system is obtained if of the two reflected types of wave only one exists, the amplitude of the other one being zero. This occurs at that value of the angle of incidence i for which the determinant of the coefficients of the 4 remaining amplitudes figuring in the 4 boundary conditions is zero.

KNOTT's calculations do not hold any longer when the amplitude of the incident wave is put equal to zero. The wave system then consists of two reflected waves and two refracted waves, while the angle i is, of course, determined again by a determinant equation. It appears that this equation is equivalent to the equation of the generalised RAYLEIGH waves derived by STONELEY[2]) in 1924.

This peculiar system of waves is seismologically of importance, because the amplitudes appear to decrease in this case in both media exponential with increasing distance to the surface of separation. Strong earthquake waves which met an interface at which these STONELEY waves are possible reach the surface of the earth very much damped and are therefore registered as weak vibrations. Consequently it is of importance to investigate at what values of the material constants of the two media a STONELEY wave system can exist, *i.e.* the STONELEY equation can be solved.

In the first part of this paper the above derivation of the STONELEY equation as an extension of the theory of KNOTT will be given; in the second part an enquiry will be made into the values of the material constants for which the STONELEY equation can be solved.

§ 2. *Derivation of the* STONELEY *equation.*

We suppose the bounding surface between the two media to be the plane $z = 0$ ($z > 0$ in medium 2) and the incident wave, being longitudinal, is propagated in medium 1. Putting the angle of incidence i_1 this wave can be expressed by $A_e . F(pt - h_1 x \sin i_1 \quad h_1 z \cos i_1)$; the remaining 4 waves are then:

the reflected longitudinal wave: $A_r . F(pt - h_1 x \sin i_1 + h_1 z \cos i_1)$
the reflected transversal wave: $\mathfrak{A}_r . F(pt - \mathfrak{k}_1 x \sin r_1 + \mathfrak{k}_1 z \cos r_1)$
the refracted longitudinal wave: $A_d . F(pt - h_2 x \sin i_2 - h_2 z \cos i_2)$
the refracted transversal wave: $\mathfrak{A}_d . F(pt - \mathfrak{k}_2 x \sin r_2 - \mathfrak{k}_2 z \cos r_2)$

where $\frac{p}{2\pi}$ = the frequency,

$h=\frac{p}{V}$, V being the phase velocity of the longitudinal waves

$\mathfrak{k}=\frac{p}{\mathfrak{V}}$, $\mathfrak{V}$ being the phase velocity of the transversal waves.

The boundary conditions are that the displacement and stress are continuous at $z=0$. We thus obtain:

$$A_e \sin i_1 + A_r \sin i_1 + \mathfrak{A}_r \cos r_1 = A_d \sin i_2 + \mathfrak{A}_d \cos i_2$$

(tangential component of the displacement)

$$A_e \cos i_1 - A_r \cos i_1 + \mathfrak{A}_r \sin r_1 = A_d \cos i_2 - \mathfrak{A}_d \sin r_2$$

(normal component of the displacement)

$$A_e \cos 2\,r_1 + A_r \cos 2\,r_1 - \mathfrak{A}_r \frac{\sin 2\,r_1}{n_1} = \frac{\varrho_2\,V_2}{\varrho_1\,V_1} A_d \cos 2\,r_2 - \frac{\varrho_2\,\mathfrak{V}_2}{\varrho_1\,V_1} \mathfrak{A}_d \sin 2\,r_2$$

(normal component of the tension)

$$A_e \sin 2\,i_1 - A_r \sin 2\,i_1 - \mathfrak{A}_r\, n_1 \cos 2\,r_1 = \frac{\mu_2\,V_1}{\mu_1\,V_2} A_d \sin 2\,i_2 + \frac{\mu_2\,V_1}{\mu_1\,\mathfrak{V}_2} \mathfrak{A}_d \cos 2\,r_2$$

(tangential component of the tension)

in which $n_1=\frac{V_1}{\mathfrak{V}_1}$, ϱ = the density and μ = the rigidity.

If we suppose the incident wave to be transversal and vibrating in the plane of inci-

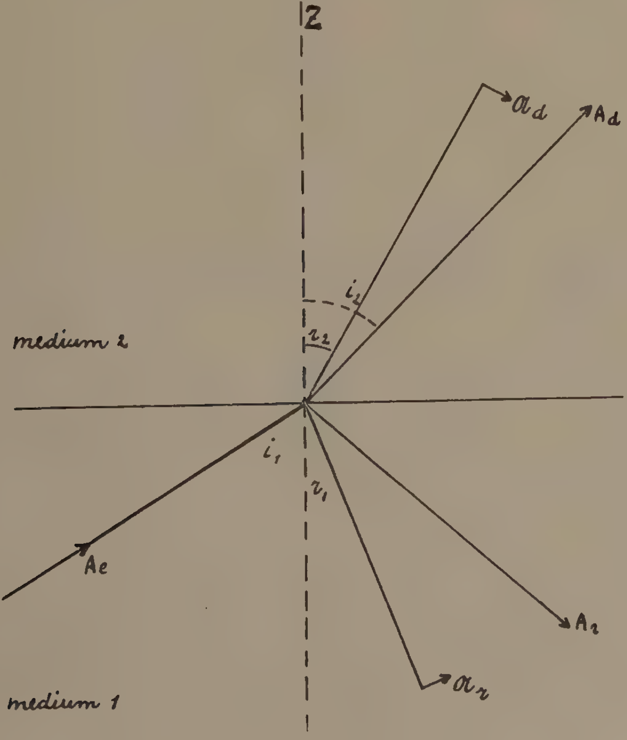

Fig. 1.

dence (figure 2), this wave can be expressed by $\mathfrak{A}_e . F\,(pt - \mathfrak{k}_1\, x \sin r_1 - \mathfrak{k}_1\, z \cos r_1)$. Writing the boundary conditions in the same order of following as above we get:

$$\mathfrak{A}_e \cos r_1 + \mathfrak{A}_r \cos r_1 + A_r \sin i_1 = \mathfrak{A}_d \cos r_2 + A_d \sin i_2$$

$$\mathfrak{A}_e \sin r_1 - \mathfrak{A}_r \sin r_1 + A_r \cos i_1 = \mathfrak{A}_d \sin r_2 - A_d \cos i_2$$

$$\mathfrak{A}_e \sin 2 r_1 + \mathfrak{A}_r \sin 2 r_1 - A_r n_1 \cos 2 r_1 = \frac{\varrho_2 \mathfrak{B}_2}{\varrho_1 \mathfrak{B}_1} \mathfrak{A}_d \sin 2 r_2 - \frac{\varrho_2 V_2}{\varrho_1 \mathfrak{B}_1} A_d \cos 2 r_2$$

$$\mathfrak{A}_e \cos 2 r_1 - \mathfrak{A}_r \cos 2 r_1 - A_r \frac{\sin 2 i_1}{n_1} = \frac{\mu_2 \mathfrak{B}_1}{\mu_1 \mathfrak{B}_2} \mathfrak{A}_d \cos 2 r_2 + \frac{\mu_2 \mathfrak{B}_1}{\mu_1 V_2} A_d \sin 2 i_2 .$$

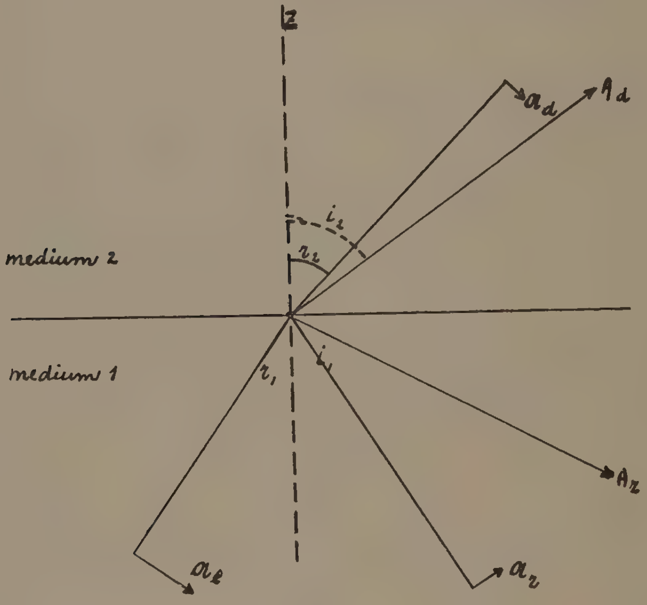

Fig. 2.

By means of such a set of equations the amplitudes of 4 waves can be expressed in the amplitude of the fifth, unless the latter is equal to zero. In that case the determinant of the coefficients of the homogeneous set of equations should be zero. If therefore we put $A_e = 0$ (fig. 1), then:

$$\begin{vmatrix} \sin i_1 & \cos r_1 & \sin i_2 & \cos r_2 \\ \cos i_1 & \sin r_1 & \cos i_2 & -\sin r_2 \\ \cos 2 r_1 & -\dfrac{\sin 2 r_1}{n_1} & \dfrac{\varrho_2 V_2}{\varrho_1 V_1} \cos 2 r_2 & -\dfrac{\varrho_2 \mathfrak{B}_2}{\varrho_1 V_1} \sin 2 r_2 \\ \sin 2 i_1 & -n_1 \cos 2 r_1 & \dfrac{\mu_2 V_1}{\mu_1 V_2} \sin 2 i_2 & \dfrac{\mu_2 V_1}{\mu_1 \mathfrak{B}_2} \cos 2 r_2 \end{vmatrix} = 0.$$

Expanding the determinant in terms of the minors of the second order formed by the first and the third row, we obtain

$$P + S + Q_1 + Q_2 + R_1 + R_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$P = n_1 n_2 \sin^2 r_2 \left(\frac{\varrho_2}{\varrho_1} \cos 2\, r_2 - \cos 2\, r_1 \right)^2,$$

$$S = 4 \cos i_1 \cos i_2 \cos r_1 \cos r_2 \sin^2 r_1 \left(\frac{\mu_2}{\mu_1} - 1 \right)^2,$$

$$Q_1 = n_1 \cos i_2 \cos r_2 \left\{ 2 \sin^2 r_1 \left(\frac{\mu_2}{\mu_1} - 1 \right) + 1 \right\}^2,$$

$$Q_2 = n_2 \cos i_1 \cos r_1 \left\{ 2 \sin^2 r_2 \left(\frac{\mu_1}{\mu_2} - 1 \right) + 1 \right\}^2 \cdot \frac{\varrho_2\, \mu_2}{\varrho_1\, \mu_1},$$

$$R_1 = \frac{\mu_2}{\mu_1} \frac{\sin i_1 \cos i_2 \cos r_1}{\sin r_2} \quad \text{and} \quad R_2 = \frac{\varrho_2}{\varrho_1} \frac{\sin i_2 \cos i_1 \cos r_2}{\sin r_1}.$$

If we multiply these terms by μ_1/ϱ_2, it will be at once obvious that P and S are symmetrical with respect to the two suffixes 1 and 2, while Q_1 and Q_2, R_1 and R_2, if we interchange these suffixes, change into each other, so that equation (1) is symmetrical with respect to the two media.

The same equation is obtained by putting the amplitude $\mathfrak{A}_e = 0$ in the second of the two cases mentioned above (fig. 2), which is evident, because we have then the same system of waves, namely $\{A_r\ \mathfrak{A}_r\ A_d\ \mathfrak{A}_d\}$ in both cases.

Other special wave systems are possible, at wich not the incident wave, but one of the reflected waves disappears. If, for instance, in case of a longitudinal incident wave $\{A_e\}$ we put the amplitude of the longitudinal reflected wave equal to zero, we get the system $\{A_e\ \mathfrak{A}_r\ A_d\ \mathfrak{A}_d\}$; reducing the determinant equation we obtain here:

$$P - S + Q_1 - Q_2 + R_1 - R_2 = 0.$$

Further it is possible that the incident wave is propagated in the second medium (such a wave will be denoted by an accent) and that then one of the reflected waves does not exist. Therefore the following cases of special reflection occur:

$\{A_e, \mathfrak{A}_r, A_d, \mathfrak{A}_d\}$; the corresponding equation being: $P - S + Q_1 - Q_2 + R_1 - R_2 = 0$
$\{\mathfrak{A}_e, A_r, A_d, \mathfrak{A}_d\}$; the corresponding equation being: $P - S + Q_1 - Q_2 - R_1 + R_2 = 0$
$\{A'_e, \mathfrak{A}_d, A_r, \mathfrak{A}_r\}$; the corresponding equation being: $P - S - Q_1 + Q_2 - R_1 + R_2 = 0$
$\{\mathfrak{A}'_e, A_d, A_r, \mathfrak{A}_r\}$; the corresponding equation being: $P - S - Q_1 + Q_2 + R_1 - R_2 = 0$

(changing the suffixes 1 and 2 the first equation is identical with the third).

These 5 equations are all irrational, as will be obvious if we express all the circular functions in say $\sin i_1$. If we attempt to make one of the equations rational, roots are generally introduced; as all terms of these equations only differ in sign these introduced roots are roots of one of the other equations. Therefore if we rationalise equation (1) the roots introduced will be the roots of the equations for special reflection; these roots are however irrelevant. In consequence if we want to determine the value of i_1, where the wave system $\{A_r, \mathfrak{A}_r, A_d, \mathfrak{A}_d\}$ is possible, we must solve equation (1) without squaring.

Now this equation can be transformed into a more comprehensible form on account of the following remarks:

1. All terms of the left-hand side are positive if the circular functions occurring in it are real. Then equation (1) has no solution.

2. If one of the sines is imaginary, then they are all so, according to SNELLIUS' law; the terms are then partly positive partly negative, so there is a solution *à priori* possible.

3. If some, but not all, of the cosines are imaginary, the coefficients of equation (1) are partly real partly imaginary. In this case too a solution is impossible. If however all cosines are imaginary then the terms are partly negative partly positive, and the equation can, therefore, be solved. Consequently the sines of all occurring angles must be greater than 1, which is certainly true if the sinus of the smallest of these angles is greater than 1.

As $\frac{\sin i_1}{\sin r_1} = \frac{V_1}{\mathfrak{V}_1}$ and $\frac{\sin i_2}{\sin r_2} = \frac{V_2}{\mathfrak{V}_2}$ and $V_1 > \mathfrak{V}_1$, $V_2 > \mathfrak{V}_2$, r_1 and r_2 will always be smaller than i_1 and i_2.

Equation (1) being symmetrical with respect to the suffixes 1 and 2, we can henceforth assume $\mathfrak{V}_2 > \mathfrak{V}_1$ without restricting the problem; then r_1 is the smallest angle. So when all cosines are imaginary $\sin r_1 > 1$.

Summarizing these above remarks, we can assert that a solution of equation (1) is only possible if $\sin r_1$ is imaginary or is greater than 1; or putting it otherwise: $\sin^2 r_1 < 0$ or $\sin^2 r_1 > 1$.

We can condense these two inequalities into the following one: $\frac{1}{\sin^2 r_1} < 1$.

Therefore it is advisable to use $\frac{1}{\sin^2 r_1}$ as a new variable, which we shall call ζ.

If

$$\sin r_1 = \frac{1}{\sqrt{\zeta}}, \text{ then } \sin i_1 = \frac{n_1}{\sqrt{\zeta}}, \ \sin i_2 = \frac{m_1}{\sqrt{\zeta}}, \ \sin r_2 = \frac{m_2}{\sqrt{\zeta}}$$

where $m_1 = \frac{V_2}{\mathfrak{V}_1}$ and $m_2 = \frac{\mathfrak{V}_2}{\mathfrak{V}_1}$.

Hence

$$\cos r_1 = \frac{i\sqrt{1-\zeta}}{\sqrt{\zeta}}, \qquad \cos i_1 = \frac{i n_1 \sqrt{1-\nu_1 \zeta}}{\sqrt{\zeta}},$$

$$\cos i_2 = \frac{i\, m_1 \sqrt{1-\alpha\zeta}}{\sqrt{\zeta}}, \qquad \cos r_2 = \frac{i\, m_2 \sqrt{1-\omega\zeta}}{\sqrt{\zeta}},$$

in which

$$\nu_1 = \frac{\mathfrak{V}_1^{\,2}}{V_1^{\,2}} = \frac{\mu_1}{\lambda_1 + 2\mu_1}, \quad \omega = \frac{\mathfrak{V}_1^{\,2}}{\mathfrak{V}_2^{\,2}} = \frac{\mu_1}{\mu_2} : \frac{\varrho_1}{\varrho_2} \text{ and}$$

$$\alpha = \frac{\mathfrak{V}_1^{\,2}}{V_2^{\,2}} = \frac{\mathfrak{V}_1^{\,2}}{\mathfrak{V}_2^{\,2}} \times \frac{\mu_2}{\lambda_2 + 2\mu_2} = \omega \nu_2$$

(λ_1 and λ_2 being the constants of incompressibility)

Equation (1) then becomes

$$\left.\begin{aligned}
&\left(2 - \frac{1-\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\zeta\right)^2 + 4\sqrt{(1-\zeta)(1-\nu_1\zeta)(1-\omega\zeta)(1-\alpha\zeta)} = \\
&\sqrt{(1-\alpha\zeta)(1-\omega\zeta)} \cdot \left(2 - \frac{1}{1-\mu_2/\mu_1}\zeta\right)^2 + \sqrt{(1-\nu_1\zeta)(1-\zeta)} \cdot \left(2 + \frac{\varrho_2/\varrho_1}{1-\mu_2/\mu_1}\zeta\right)^2 + \\
&+ \frac{\varrho_2/\varrho_1}{(1-\mu_2/\mu_1)^2}\zeta^2 \cdot \left(\sqrt{(1-\zeta)(1-\alpha\zeta)} + \sqrt{(1-\omega\zeta)(1-\nu_1\zeta)}\right).
\end{aligned}\right\} \quad (2)$$

Writing the equation derived by STONELEY in this notation we get:

$$V_r^4 \{(\varrho_1-\varrho_2)^2-(\varrho_1 x_2+\varrho_2 x_1)(\varrho_1 y_2+\varrho_2 y_1)\} +$$
$$+4\, V_r^2 \{\varrho_1 x_2 y_2-\varrho_2 x_1 y_1-(\varrho_1-\varrho_2)\} + 4\,(\mu_1-\mu_2)^2 (x_1 y_1-1)(x_2 y_2-1)=0$$

where

$$x_1=\sqrt{1-\nu_1 \frac{V_r^2}{\mathfrak{B}_1^2}},\quad x_2=\sqrt{1-a\frac{V_r^2}{\mathfrak{B}_1^2}},\quad y_1=\sqrt{1-\frac{V_r^2}{\mathfrak{B}_1^2}},\quad y_2=\sqrt{1-\omega\frac{V_r^2}{\mathfrak{B}_1^2}},$$

V_r being the phase velocity of the STONELEY waves. Putting $\frac{V_r^2}{\mathfrak{B}_1{}^2}=\zeta$ this equation is identical with equation (2).

This equation reduces to a very simple form if we take

1°. $\varrho_2=0$: $$\left(2-\frac{\zeta}{1-\mu_2/\mu_1}\right)^2=4\sqrt{(1-\nu_1\zeta)(1-\zeta)}\,;$$

with $\mu_2=0$ this is the RAYLEIGH equation.

2°. $\mu_2=0$: $$(2-\zeta)^2=4\sqrt{(1-\nu_1\zeta)(1-\zeta)}-\frac{\varrho_2}{\varrho_1}\zeta\sqrt{\frac{1-\nu_1\zeta}{1-a\zeta}}.$$

3°. $V_1=V_2$ and $\mathfrak{B}_1=\mathfrak{B}_2$ (WIECHERTS' medium):

$$\left\{(2-\zeta)-2\sqrt{(1-\nu_1\zeta)(1-\zeta)}\right\}^2=\left(\frac{1+\mu_2/\mu_1}{1-\mu_2/\mu_1}\,\zeta\right)^2.\sqrt{(1-\zeta)(1-\nu_1\zeta)}.$$

Mathematics. — *Zur Theorie der verallgemeinerten* PFAFF'*schen Gleichungen.* Von W. VAN DER KULK. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of December 27, 1941.)

Einleitung.

Ein System von Gleichungen

$$\overset{i}{F}(\xi^{\varkappa}, d_1\,\xi^{[\mu_1}\ldots d_{\mathrm{m}}\,\xi^{\mu_m]}) = 0\,;\ i = \mathrm{d}+1, \ldots, \mathrm{m}\,(\mathrm{n}-\mathrm{m}),$$

wo die $\overset{i}{F}$ in den $\binom{n}{m}$ Ausdrücken $d_1\xi^{[\mu_1}\ldots d_{\mathrm{m}}\,\xi^{\mu_m]}$ homogen aber übrigens beliebig sind, heisst ein System von verallgemeinerten PFAFF'schen Gleichungen. Es wird nun ein Theorem angekündigt, das für diese verallgemeinerten Systeme dieselbe Rolle spielt wie das bekannte CARTAN-KÄHLER'sche Theorem für die in den $d_1\,\xi^{[\mu_1}\ldots d_{\mathrm{m}}\,\xi^{\mu_m]}$ homogenen und *linearen* PFAFF'schen Gleichungen. Für $m = 2$ wurde das Theorem schon früher bewiesen [1]), für $m > 2$ soll der Beweis an anderer Stelle veröffentlicht werden. Für $m = n - 1$ ist das Theorem gleichbedeutend mit dem bekannten JACOBI'schen Theorem über die homogenen partiellen Differentialgleichungen in einer Unbekannten.

1. *Das $\mathfrak{S}^m_d$-Feld und die zugehörigen verallgemeinerten* PFAFF'*schen Systeme.*

Eine m-Richtung in einem beliebigen Punkte $\xi^{\varkappa}$; $\varkappa = 1, \ldots, n$ eines n-dimensionalen Raumes X_n ist eine von m von $\xi^{\varkappa}$ ausgehenden Linienelementen $d_1\,\xi^{\varkappa}, d_2\,\xi^{\varkappa}, \ldots, d_{\mathrm{m}}\,\xi^{\varkappa}$ aufgespannte m-dimensionale Hyperebene in dem Lokalraum von $\xi^{\varkappa}$. Eine solche m-Richtung lässt sich bekanntlich festlegen durch die $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ in den Indizes $\varkappa_1, \ldots, \varkappa_m$ alternierenden GRASSMANN'schen Koordinaten $v^{\varkappa_1 \ldots \varkappa_m} = d_1\,\xi^{[\mu_1}\ldots d_{\mathrm{m}}\,\xi^{\mu_m]}$; $\varkappa_1, \ldots, \varkappa_m = 1, \ldots, n$, zwischen denen die PLÜCKER'schen Relationen

$$v^{[\varkappa_1 \ldots \varkappa_m}\, v^{\mu_1] \ldots \mu_m} = 0\,;\ \varkappa_1, \ldots, \varkappa_m, \mu_1, \ldots, \mu_m = 1, \ldots, n, \quad . \quad (1.\,1)$$

existieren. In jedem Punkte $\xi^{\varkappa}$ gibt es $\infty^{m(n-m)}$ solcher m-Richtungen. Ordnet man dem Punkte $\xi^{\varkappa}$ einen projektiven, $\left\{\binom{n}{m} - 1\right\}$-dimensionalen Raum $\mathfrak{P}^m$ zu (die $\binom{n}{m}$ Koordinaten in diesem Raume seien mit $v^{\varkappa_1 \ldots \varkappa_m}$ bezeichnet, wo $v^{\varkappa_1 \ldots \varkappa_m}$ in $\varkappa_1, \ldots, \varkappa_m$ alternierend ist), so werden diese $\infty^{m(n-m)}$ m-Richtungen in $\mathfrak{P}^m$ auf die Punkte einer $m\,(n-m)$-dimensionalen, irreduziblen, algebraischen Mannigfaltigkeit $\mathfrak{S}^m$, der s.g. GRASSMANN'schen Mannigfaltigkeit, mit den Gleichungen (1. 1) algebildet [2]).

Die $\{m\,(n-m) - d\}$ von (1. 1) unabhängigen Gleichungen in den $v^{\mu_1 \ldots \mu_m}$ als Unbekannten und den $\xi^{\varkappa}$ als Parametern

$$\overset{i}{F}(\xi^{\varkappa}, v^{\mu_1 \ldots \mu_m}) = 0\,;\ i = \mathrm{d}+1, \ldots, \mathrm{m}\,(\mathrm{n}-\mathrm{m}), \quad . \quad . \quad . \quad (1.\,2)$$

mit in den $\xi^{\varkappa}$ und $v^{\mu_1 \ldots \mu_m}$ analytischen und in den $v^{\mu_1 \ldots \mu_m}$ ausserdem noch homogenen Funktionen $\overset{i}{F}$ (die übrigens aber beliebig sind), bestimmen also zusammen mit (1. 1) in jedem Punkte $\xi^\varkappa$ der X_n ∞^d m-Richtungen, die in der lokalen $\mathfrak{P}^m$ von $\xi^\varkappa$ mit den Punkten einer d-dimensionalen Teilmannigfaltigkeit $\mathfrak{S}^m_d$ von $\mathfrak{S}^m$ korrespondieren. In dieser Weise entsteht in der X_n ein s.g. $\mathfrak{S}^m_d$-Feld. Damit in einer Umgebung einer Nullstelle $\xi^\varkappa$, $v^{\mu_1\dots\mu_m}$ ($v^{\mu_1\dots\mu_m} \neq 0$) von (1. 1) und (1. 2) die Gleichungen (1. 2) wirklich von den Gleichungen (1. 1) unabhängig sind, ist hinreichend, dass für diese Nullstelle die linearen Formen

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[1]) Siehe: Eine Verallgemeinerung eines Theorems aus der Theorie der PFAFF'schen Gleichungen für den einfachsten Fall $m = 2$, I und II, Proc. Ned. Akad. v. Wet., Amsterdam, **44**, 452—463 und 625—635 (1941), hier weiter zitiert als V.P. I und II. Für Literaturangaben vgl. V.P. I.

[2]) Für $m = 1$ wird (1. 1) eine Identität. In $\mathfrak{P}^1$ ist also $\mathfrak{S}^1 = \mathfrak{P}^1$.

$$\overset{i}{F}_{\mu\mu_2\dots\mu_m} v^{\lambda\mu_2\dots\mu_m} Z^\mu_\lambda\,;\ \overset{i}{F}_{\mu_1\dots\mu_m} \overset{\text{def}}{=} \frac{\partial \overset{i}{F}}{\partial v^{\mu_1\dots\mu_m}}\,;\ i = \mathrm{d}+1, \dots, \mathrm{m}(\mathrm{n}-\mathrm{m}), \quad (1.\ 3)$$

in den n^2 Variablen Z^μ_λ; $\mu, \lambda = 1, \dots, n$, linear unabhängig sind. Eine solche Nullstelle heisse *regulär*. Der d-dimensionale Tangentialraum der lokalen $\mathfrak{S}^m_d$ von $\xi^\varkappa$ im Punkte $v^{\mu_1\dots\mu_m}$ besteht aus allen m-Vektoren von der Form $Z^{[\mu_1}_\lambda v^{|\lambda|\mu_2\dots\mu_m]}$, wo Z^μ_λ die Formen (1. 3) annulliert [3]). Im folgenden werden nur reguläre Nullstellen in Betracht gezogen.

Das System von Gleichungen

$$\overset{i}{F}(\xi^\varkappa, d_1\, \xi^{[\mu_1} \dots d_{\mathrm{m}}\, \xi^{\mu_m]}) = 0\,;\ i = \mathrm{d}+1, \dots, \mathrm{m}(\mathrm{n}-\mathrm{m}), \quad (1.\ 4)$$

wo $d_1\,\xi^\varkappa, \dots, d_{\mathrm{m}}\,\xi^\varkappa$ unabhängige Differentiale sind, möge ein dem $\mathfrak{S}^m_d$-Felde *adjungiertes System* von verallgemeinerten PFAFF'schen Gleichungen heissen. Es ist klar, dass jedes $\mathfrak{S}^m_d$-Feld unendlich viele adjungierte Systeme besitzt, die jedes für sich das $\mathfrak{S}^m_d$-Feld eindeutig festlegen.

Jede m-Richtung in einem Punkte $\xi^\varkappa$ enthält ∞^{m-1} $(m-1)$-Richtungen, die in der lokalen $\mathfrak{P}^{m-1}$ von $\xi^\varkappa$ auf die Punkte eines linearen, $(m-1)$-dimensionalen Raumes abgebildet werden. Mit ∞^d m-Richtungen korrespondieren in der $\mathfrak{P}^{m-1}$ also ∞^d lineare, $(m-1)$-dimensionale Räume, Daraus folgt, dass ein $\mathfrak{S}^m_d$-Feld ein $\mathfrak{S}^{m-1}_{d_{m-1}}$-Feld induziert, dessen lokale $\mathfrak{S}^{m-1}_{d_{m-1}}$ in jedem Punkte $\xi^\varkappa$ aus ∞^d linearen, $(m-1)$-dimensionalen Räumen besteht. Für die Dimension d_{m-1} gilt die Ungleichung

$$m-1+\frac{d}{m} \leqq d_{m-1} \leqq m-1+d. \quad (1.\ 5)$$

Dieses $\mathfrak{S}^{m-1}_{d_{m-1}}$-Feld induziert in ähnlicher Weise ein $\mathfrak{S}^{m-2}_{d_{m-2}}$-Feld, das selbst wieder ein $\mathfrak{S}^{m-3}_{d_{m-3}}$-Feld induziert. So weiterfahrend, lässt sich für jeden Wert von r von 1 bis $m-1$ ein $\mathfrak{S}^r_{d_r}$-Feld konstruieren, dessen lokale $\mathfrak{S}^r_{d_r}$ in jedem Punkte $\xi^\varkappa$ aus $\infty^{d_{r+1}}$ in der lokalen $\mathfrak{P}^r$ liegenden, linearen, r-dimensionalen Räumen besteht (für d_m ist dabei d zu nehmen). Diese lokale $\mathfrak{S}^r_{d_r}$ ist auch die Menge aller in den ∞^d m-Richtungen der lokalen $\mathfrak{S}^m_d$ des ursprünglichen Feldes enthaltenen r-Richtungen. Im folgenden werden immer nur die regulären Nullstellen dieses $\mathfrak{S}^r_{d_r}$-Feldes betrachtet.

2. *Vollständige Integrabilität und Vollständigkeit.*

Eine m-dimensionale Fläche X_m in der X_n lässt sich in der Parameterform

$$\xi^\varkappa = f^\varkappa(\eta^a)\,;\ \varkappa = 1, \dots, n\,;\ a = 1, \dots, \mathrm{m}, \quad (2.\ 1)$$

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3) Vgl. V.P. I, S 212 u.f. Da ist alles für $m = 2$ bewiesen.

schreiben, wo die Funktionen $f^{\varkappa}$ in den η^a analytisch sind und die Matrix der

$$B_a^{\varkappa} \overset{\text{def}}{=} \partial_a f^{\varkappa}\,;\ \partial_a \overset{\text{def}}{=} \frac{\partial}{\partial \eta^a}\,;\ \varkappa = 1, \ldots, n\,;\ a = 1, \ldots, \mathrm{m}, \quad . \quad . \qquad (2.\,2)$$

genau den Rang m hat. Die tangierende m-Richtung in einem Punkte $\xi^{\varkappa}$ der X_m hat die GRASSMANN'schen Koordinaten

$$v^{\mu_1 \ldots \mu_m} = B_1^{[\mu_1} \ldots B_{\mathrm{m}}^{\mu_m]} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.\,3)$$

Gehört diese m-Richtung in jedem Punkte $\xi^{\varkappa}$ der X_m der lokalen $\mathfrak{S}_d^m$ eines $\mathfrak{S}_d^m$-Feldes an, so nennt man die X_m eine *Integral-X_m* dieses Feldes, oder auch eine Integral-X_m der dem Felde adjungierten verallgemeinerten PFAFF'schen Systeme. Analytisch bedeutet dies, dass für jeden Punkt $\xi^{\varkappa}$ einer solchen X_m das Wertsystem $\xi^{\varkappa}$, $B_1^{[\mu_1} \ldots B_{\mathrm{m}}^{\mu_m]}$ eine Nullstelle von einem (und also auch von jedem) der adjungierten verallgemeinerten PFAFF'schen Systeme des Feldes ist. Jede in einer Integral-X_m eines $\mathfrak{S}_d^m$-Feldes liegende X_r ist Integral-X_r des von diesem Felde induzierten $\mathfrak{S}_{d_r}^r$-Feldes.

Gibt es zu jeder Nullstelle $\xi^{\varkappa}$, $v^{\mu_1 \ldots \mu_m}$ eines $\mathfrak{S}_d^m$-Feldes[4]) mindestens eine Integral-X_m, die den Punkt $\xi^{\varkappa}$ enthält, und dort die m-Richtung $v^{\mu_1 \ldots \mu_m}$ tangiert, so heissen das Feld und die dem Felde adjungierten verallgemeinerten PFAFF'schen Systeme *vollständig integrabel*. Die induzierten Felder eines solchen Feldes sind ebenfalls vollständig integrabel. Man kann nun folgenden Satz beweisen:

Notwendig für die vollständige Integrabilität eines $\mathfrak{S}_d^m$-Feldes mit einem adjungierten System (1. 4) *ist, dass für jede Nullstelle $\xi^{\varkappa}$, $v^{\mu_1 \ldots \mu_m}$ des Feldes die Gleichungen*

$$\left.\begin{aligned} v^{\omega \varrho_2 \ldots \varrho_m} \{ \bar{\partial}_\omega \overset{i}{F} + m \overset{i}{F}_{\mu \mu_2 \ldots \mu_m} v^{\lambda \mu_2 \ldots \mu_m} Z_{\omega \lambda}^{\mu} \} = 0\,; \\ i = \mathrm{d} + 1, \ldots, \mathrm{m}\,(\mathrm{n} - \mathrm{m}), \\ \varrho_2, \ldots, \varrho_m = 1, \ldots, n \end{aligned}\right\} \quad . \quad . \qquad (2.\,4)$$

mit den $\frac{1}{2} n^2 (n+1)$ *Unbekannten* $Z_{\mu\lambda}^{\varkappa}$; $Z_{\mu\lambda}^{\varkappa} = Z_{\lambda\mu}^{\varkappa}$; $\varkappa, \mu, \lambda = 1, \ldots, n$, *wo*

$$\bar{\partial}_\omega \overset{i}{F} \overset{\text{def}}{=} \frac{\partial \overset{i}{F}}{\partial \xi^\omega} \quad (v^{\mu_1 \ldots \mu_m} \ \textbf{nicht}\ \textit{differenzieren}) \quad . \quad . \quad . \quad . \qquad (2.\,5)$$

bedeutet, mindestens eine Lösung hat.

Die Lösbarkeit des Systems (2. 4) ändert sich nicht, wenn man die $\overset{i}{F}$ durch die Funktionen $\overset{i'}{F}$ eines anderen adjungierten Systems des Feldes ersetzt. Die Lösbarkeit von (2. 4) für eine Nullstelle $\xi^{\varkappa}$, $v^{\mu_1 \ldots \mu_m}$ bedeutet geometrisch, dass mindestens eine X_m existiert, die die m-Richtung $v^{\mu_1 \ldots \mu_m}$ im Punkte $\xi^{\varkappa}$ tangiert und die ausserdem in einer infinitesimalen Umgebung von $\xi^{\varkappa}$ in zweiter Annäherung eine Integral-X_m des Feldes ist.

Ein $\mathfrak{S}_d^m$-Feld, das die im obigen Satze erwähnte notwendige Bedingung erfüllt, heisse *vollständig*. Aus der obenerwähnten geometrischen Deutung folgt, dass für ein solches Feld auch die vom Felde induzierten Felder alle vollständig sind. Die Vollständigkeit eines $\mathfrak{S}_d^m$-Feldes ist zwar notwendig, aber nicht hinreichend für die vollständige Integrabilität[5]). Es fehlt für die vollständige Integrabilität eines vollständigen Feldes also noch eine passende Bedingung, die im folgenden Paragraphen entwickelt werden soll.

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[4]) Eine Nullstelle von (1. 1) und (1. 2) heisst eine Nullstelle des von (1. 1) und (1. 2) bestimmten $\mathfrak{S}_d^m$-Feldes.

[5]) Für ein Gegenbeispiel siehe V.P. II, S. 627. Da findet sich auch ein Beweis des oben formulierten Satzes.

3. *Die stationären Felder.*

Die lokale $\mathfrak{S}^{m-1}_{d_{m-1}}$ des von einem $\mathfrak{S}^{m}_{d}$-Felde induzierten $\mathfrak{S}^{m-1}_{d_{m-1}}$-Feldes in einem Punkte $\xi^{\varkappa}$ besteht, wie erwähnt wurde, aus ∞^{d} linearen $(m-1)$-dimensionalen Räumen. Es sei L_{m-1} ein beliebiger dieser Räume und es sei $P_1, \dots, P_m$ ein beliebiger Punkt-m-Tüpel in diesem Raume ($P_1, \dots, P_m$ seien linear unabhängig). Die Tangentialräume der lokalen $\mathfrak{S}^{m-1}_{d_{m-1}}$ in diesen m Punkten seien mit $T_1, \dots, T_m$ bezeichnet und der Tangentialraum der ganzen $\mathfrak{S}^{m-1}$ im Punkte P_1 mit S_1. S_1 enthält T_1. Ist nun Q_α ein beliebiger Punkt in T_α; $\alpha = 2, \dots, m$, so heisst der Verbindungsraum von $Q_2, \dots, Q_m$, der also aus allen Punkten von der Form $\lambda^{\alpha} Q_\alpha$ mit beliebigen λ_α; $\alpha = 2, \dots, m$ besteht, eine *Transversale* von $T_2, \dots, T_m$. Der Punkt-m-Tüpel $P_1, \dots, P_m$ heisse nun *stationär*, wenn für jede Transversale von $T_2, \dots, T_m$ der Schnittraum mit S_1 (falls dieser Schnittraum nicht leer ist) ganz in T_1 enthalten ist. Ist jeder Punkt-m-Tüpel in der L_{m-1} stationär, so wird diese L_{m-1} *stationär* genannt. Hinreichend dazu ist schon, dass alle Punkt-m-Tüpel $P_1, \dots, P_m$ mit festgewählten Punkten $P_2, \dots, P_m$ aber mit freiem Punkte P_1 stationär sind. Ist jede L_{m-1} von jeder lokalen $\mathfrak{S}^{m-1}_{d_{m-1}}$ des $\mathfrak{S}^{m-1}_{d_{m-1}}$-Feldes stationär, so heisst auch dieses Feld *stationär*. Da das vom $\mathfrak{S}^{m}_{d}$-Felde induzierte $\mathfrak{S}^{r}_{d_r}$-Feld auch vom induzierten $\mathfrak{S}^{r+1}_{d_{r+1}}$-Feld induziert wird, lässt sich der Begriff „stationär" für dieses Feld in ähnlicher Weise erklären ($r = 1, \dots, m-1$).

Betrachten wir insbesondere den Fall, wo das induzierte $\mathfrak{S}^{1}_{d_1}$-Feld stationär ist. Die lokale $\mathfrak{S}^{1}_{d_1}$ im Punkte $\xi^{\varkappa}$ besteht in diesem Falle aus ∞^{d_2} linearen, 1-dimensionalen Räumen, d.h. aus ∞^{d_2} Geraden in $\mathfrak{P}^1$. Die $\mathfrak{S}^{1}_{d_1}$ ist in $\mathfrak{P}^1$ also eine Regelfläche 6). Es sei L eine dieser Geraden und es seien P_1 und P_2 zwei beliebige Punkte auf L. Die Tangentialräume der $\mathfrak{S}^{1}_{d_1}$ in P_1 und P_2 seien T_1 und T_2. Die Transversalen von T_2 sind die Punkte in T_2. Da $\mathfrak{S}^1 = \mathfrak{P}^1$ ist (vgl. Fussnote 2), fällt der Tangentialraum S_1 der $\mathfrak{S}^1$ im Punkte P_1 mit $\mathfrak{P}^1$ zusammen und daraus geht hervor, dass die Schnitträume dieser Transversalen mit dem Tangentialraum S_1 die Transversalen selbst sind. Da P_1, P_2 ein stationärer Punkt-2-Tüpel ist, muss also T_2 ganz in T_1 enthalten sein. T_1 und T_2 sind aber beide d_1-dimensional und fallen somit zusammen. Die lokale $\mathfrak{S}^{1}_{d_1}$ in jedem Punkte $\xi^{\varkappa}$ hat also längs jeder ihrer ∞^{d_2} Geraden einen unveränderlichen Tangentialraum, d.h. diese $\mathfrak{S}^{1}_{d_1}$ ist ***abwickelbar***.

4. *Hinreichende Bedingungen für die vollständige Integrabilität.*

Jetzt lässt sich das angekündigte Theorem leicht formulieren:

Ein vollständiges $\mathfrak{S}^{m}_{d}$-Feld, dessen induziertes $\mathfrak{S}^{r}_{d_r}$-Feld für jeden Wert von r von 1 bis zu $m-1$ stationär ist, ist vollständig integrabel 7).

Dieses Theorem lässt sich durch Hinzufügung solcher Anfangsbedingungen, die eine Integral-X_m eindeutig festlegen, ergänzen.

Die im zweiten Paragraphen noch fehlende Bedingung, die die vollständige Integrabilität eines vollständigen $\mathfrak{S}^{m}_{d}$-Feldes sicher stellt, besagt also, dass alle induzierten Felder des gegebenen $\mathfrak{S}^{m}_{d}$-Feldes stationär sein müssen. Diese Bedingung ist zusammen mit der Vollständigkeit zwar hinreichend, jedoch nicht notwendig für die vollständige Integrabilität, denn es gibt vollständig integrabele $\mathfrak{S}^{m}_{d}$-Felder, deren induzierte Felder dennoch nicht stationär sind.

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6) In V. P. I und II wurde diese Regelfläche mit $\mathfrak{R}_t$ statt mit $\mathfrak{S}^{1}_{d_1}$ bezeichnet.

7) Für $m = 2$ geht dieses Theorem in das in V.P. II bewiesene Haupttheorem über.

Vergleicht man dieses Theorem mit dem CARTAN-KÄHLER'schen, so ist u.a. folgendes zu bemerken. Die bei KÄHLER auftretenden $\mathfrak{S}_d^m$-Felder haben mindestens ein adjungiertes PFAFF'sches System mit in den $v^{\mu_1 \dots \mu_m}$ linearen Funktionen $\overset{i}{F}$, d.h. die lokale $\mathfrak{S}_d^m$ eines solchen Feldes ist in der $\mathfrak{P}^m$ von jedem Punkte $\xi^\varkappa$ Durchschnitt von $\mathfrak{S}^m$ mit einem linearen Raum. Auch müssen alle induzierten Felder bei KÄHLER diese Eigenschaft besitzen. Man überzeugt sich leicht davon, dass die induzierten Felder mit dieser Eigenschaft auch stationär sind. Im obigen Theorem brauchen aber weder das gegebene noch die induzierten Felder in jedem Punkte $\xi^\varkappa$ Durchschnitte der $\mathfrak{S}^m$ (bzw. der $\mathfrak{S}^r$) mit linearen Räumen zu sein. Nur müssen die induzierten Felder alle stationär sein. Die Funktionen $\overset{i}{F}$ in diesem Theorem brauchen also nicht linear in den $v^{\mu_1 \dots \mu_m}$ zu sein, doch können sogar transzendent in den $v^{\mu_1 \dots \mu_m}$ sein[8]).

Schliesslich sei folgender bemerkenswerter Satz erwähnt:

Ein vollständiges $\mathfrak{S}_d^m$-Feld, dessen induziertes $\mathfrak{S}_{d_1}^1$-Feld stationär (d.h. abwickelbar) ist und die Dimension $d_1 = m$ hat, ist vollständig integrabel[9]).

Denn es lässt sich beweisen, dass die induzierten Felder eines $\mathfrak{S}_d^m$-Feldes alle stationär sind, sobald das induzierte $\mathfrak{S}_{d_1}^1$-Feld stationär (d.h. abwickelbar) ist und die Dimension $d_1 = m$ hat.

Für $m = n - 1$ und $d \geqq 1$ muss $d_1 = n - 1$ sein. Denn jede $(n-1)$-Richtung in einem Punkte $\xi^\varkappa$ induziert in der lokalen $\mathfrak{P}^1$ von $\xi^\varkappa$ eine $(n-2)$-dimensionale Hyperebene. Eine $\mathfrak{S}_d^{n-1}$ in $\xi^\varkappa$, mit $d \geqq 1$, induziert in dieser $\mathfrak{P}^1$ also eine $\mathfrak{S}_{d_1}^1$, die aus ∞^d solcher Hyperebenen besteht. Daraus folgt $d_1 \geqq n-1$. Da aber $\mathfrak{P}_1$ gerade die Dimension $n-1$ hat, ist auch $d_1 \leqq n-1$. Es ist somit $d_1 = n-1$. d.h. die $\mathfrak{S}_{d_1}^1$ ist in $\mathfrak{P}^1$ ein $(n-1)$-dimensionales Gebiet. Insbesondere ist die $\mathfrak{S}_{d_1}^1$ also auch abwickelbar. Infolge des vorhergehenden Satzes gilt daher:

Ein vollständiges $\mathfrak{S}_d^{n-1}$-Feld ist vollständig integrabel.

(Für $d = 0$ ist dieser Satz auch richtig. Denn ein $\mathfrak{S}_0^{n-1}$-Feld ist ein $(n-1)$-Richtungsfeld, und aus der Vollständigkeit eines solchen Feldes folgt leicht, dass dieses Feld X_{n-1}-bildend, d h. vollständig integrabel ist).

Dieser Satz ist gleichbedeutend mit dem JACOBI'schen Theorem der partiellen Differentialgleichungen in einer Unbekannten, ausgesprochen für den Sonderfall, wo die Gleichungen homogen in den Ableitungen dieser Unbekanten sind. Mit einem solchen System

$$\overset{i}{G}(\xi^\varkappa, \partial_\lambda s) = 0; \quad \partial_\lambda \overset{\text{def}}{=} \frac{\partial}{\partial \xi^\lambda}; \quad i = \mathrm{d}+1, \dots, \mathrm{n}-1 \quad . \quad . \quad (4.1)$$

wo die $\overset{i}{G}$ also homogen in den $\partial_\lambda s$; $\lambda = 1, \dots, n$, aber übrigens beliebig sind, ist nämlich das $\mathfrak{S}_d^{n-1}$-Feld mit den Gleichungen

$$\overset{i}{G}(\xi^\varkappa, v^{\varkappa_1 \dots \varkappa_{n-1}} e_{\varkappa_1 \dots \varkappa_{n-1} \lambda}) = 0; \quad i = \mathrm{d}+1, \dots, \mathrm{n}-1 \quad . \quad . \quad (4.2)$$

verbunden (In dieser Formel bedeutet $e_{\lambda_1 \dots \lambda_n}$ ein beliebiger kovarianter n-Vektor. Die GRASSMANN'schen Koordinaten $v^{\varkappa_1 \dots \varkappa_{n-1}}$ der $(n-1)$-Richtung eines kovarianten Vektors

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[8]) Eine ausführliche Diskussion der beiden Theoreme findet sich in V.P. I, Einleitung.

[9]) Die in V. P. I, Einleitung, S. 457 aufgestellte Behauptung, es wäre ein vollständiges $\mathfrak{S}_2^3$-Feld, dessen $\mathfrak{S}_{d_1}^1$-Feld abwickelbar ist, vollständig integrabel, ist unrichtig.

w_λ genügen den Beziehungen $v^{\varkappa_1 \dots \varkappa_{n-1}} e_{\varkappa_1 \dots \varkappa_{n-1} \lambda} = w_\lambda$. Daraus folgt tatsächlich, dass (4. 2) einen mit (4. 1) verbundenen $\mathfrak{S}_d^{n-1}$-Feld festlegt). Dieses $\mathfrak{S}_d^{n-1}$-Feld ist dann und nur dann vollständig, d.h. die Gleichungen (2. 4) sind dann und nur dann lösbar, wenn die POISSON'schen Klammerausdrücke $[\overset{i}{G}, \overset{l}{G}]$ auf dem $\mathfrak{S}_d^{n-1}$-Felde, d.h. modulo den $\overset{k}{G}$, verschwinden. Infolge des ebenformulierten Satzes ist somit dass $\mathfrak{S}_d^{n-1}$-Feld (4. 2), oder auch das System von partiellen Differentialgleichungen (4. 1), vollständig integrabel, sobald diese POISSON'schen Klammersymbole modulo den $\overset{k}{G}$; $k = \mathrm{d} + 1, \dots, \mathrm{n} - 1$ identisch verschwinden und dies ist gerade der JACOBI'sche Satz.

Das im Anfang dieses Paragraphen aufgestellte Theorem lässt sich also auch auffassen als eine Erweiterung des JACOBI'schen Theorems für gewisse Systeme von partiellen Differentialgleichungen mit einer beliebigen Anzahl von Unbekannten. Diese Systeme haben die Form

$$\overset{i}{G}\left(\xi^\varkappa, (\partial_{[\lambda_{m+1}} \overset{\mathrm{m+1}}{s}) \dots (\partial_{\lambda_n]} \overset{\mathrm{n}}{s})\right) = 0; \; i = \mathrm{d} + 1, \dots \mathrm{m}\,(\mathrm{n} - \mathrm{m}), \qquad (4.\,3)$$

wo $\overset{\mathrm{m+1}}{s}, \dots, \overset{\mathrm{n}}{s}$ die $n-m$ Unbekannten sind, und wo jede der $\overset{i}{G}$ homogen ist in den $\binom{n}{m}$ Ausdrücken $(\partial_{[\lambda_{m+1}} \overset{\mathrm{m+1}}{s}) \dots (\partial_{\lambda_n]} \overset{\mathrm{n}}{s})$; $\lambda_{m+1}, \dots, \lambda_n = 1, \dots, n$. Dem System (4. 3) lässt sich nämlich das $\mathfrak{S}_d^m$-Feld mit den Gleichungen

$$\overset{i}{G}\left(\xi^\varkappa, v^{\varkappa_1 \dots \varkappa_m} e_{\varkappa_1 \dots \varkappa_m \lambda_{m+1} \dots \lambda_n}\right) = 0; \; i = \mathrm{d} + 1, \dots, \mathrm{m}\,(\mathrm{n} - \mathrm{m}), \qquad (4.\,4)$$

zuordnen und auf diesem Felde ist das im Anfang dieses Paragraphen formulierte Theorem anwendbar. Es liegt auf der Hand, die Theorie der PFAFF'schen Gleichungen nun auch für solche Systeme, die entstehen durch Umschreibung eines Systems (4. 3) mit *nicht* homogenen Gleichungen zu entwicklen. Um aus einem solchen System (4. 3) ein PFAFF'sches System von invarianter Gestalt zu erhalten, ist es notwendig statt $e_{\lambda_1 \dots \lambda_n}$ die kovariante n-Vektordichte $\mathfrak{e}_{\lambda_1 \dots \lambda_n}$ vom Gewichte -1 zu verwenden.[10] Es entsteht dann ein System von PFAFF'schen Gleichungen von der Form

$$\overset{i}{F}\left(\xi^\varkappa, \mathfrak{B}^{\varkappa_1 \dots \varkappa_m}\right) = 0; \; i = \mathrm{d} + 1, \dots, \mathrm{m}\,(\mathrm{n} - \mathrm{m}), \quad . \quad . \quad . \qquad (4.\,5)$$

wo die $\overset{i}{F}$ *nicht* homogen in den $\binom{n}{m}$ Bestimmungszahlen der einfachen m-Vektordichte vom Gewichte $+1$ $\mathfrak{B}^{\varkappa_1 \dots \varkappa_m}$ zu sein brauchen. $\mathfrak{B}^{\varkappa_1 \dots \varkappa_m}$ genügt der Formel

$$\mathfrak{B}^{\varkappa_1 \dots \varkappa_m} \mathfrak{e}_{\varkappa_1 \dots \varkappa_m \lambda_{m+1} \dots \lambda_n} = (\partial_{[\lambda_{m+1}} \overset{\mathrm{m+1}}{s}) \dots \partial_{\lambda_n]} \overset{\mathrm{n}}{s} \quad . \quad . \quad . \qquad (4.\,6)$$

und es wäre also bei der Umgestaltung von (4. 3)

$$\overset{i}{F}\left(\xi^\varkappa, \mathfrak{B}^{\varkappa_1 \dots \varkappa_m}\right) \overset{\mathrm{def}}{=} \overset{i}{G}\left(\xi^\varkappa, \mathfrak{B}^{\varkappa_1 \dots \varkappa_m} \mathfrak{e}_{\varkappa_1 \dots \varkappa_m \lambda_{m+1} \dots \lambda_n}\right) \quad . \quad . \quad . \qquad (4.\,7)$$

zu nehmen.

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[10] Vgl. SCHOUTEN-STRUIK: Einf. i. d. neueren Meth. der Differentialgeometrie I, S. 29.

Mathematics. — *Stark konvergente Entwicklungen für die vollständigen elliptischen Integrale erster und zweiter Art.* IV. Von S. C. VAN VEEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 27, 1941.)

§ 4. *Erweiterung der vorhergehenden Ergebnisse für komplexe Werte von k.* Bisher haben wir vorausgesetzt, dass k nur reelle Werte mit $0 \leqq k < 1$ durchläuft. *Von jetzt an wird k beliebig komplex vorausgesetzt.*

Die beiden Integrale

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$$

und

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2 \sin^2 \varphi}\, d\varphi$$

sind analytische Funktionen von k im Innern der von $k = +1$ nach $+\infty$ und von $k = -1$ nach $-\infty$ aufgeschlitzten komplexen k-Ebene.

Sie sind in diesem Gebiet eindeutig bestimmt durch die Verabredung

$$|\arg(1-k \sin \varphi)| < \pi$$

und

$$|\arg(1+k \sin \varphi)| < \pi,$$

also

$$\left|\arg \sqrt{1-k^2 \sin^2 \varphi}\right| = \tfrac{1}{2} \left|\arg(1-k \sin \varphi) + \arg(1+k \sin \varphi)\right| < \pi.$$

Für $k \to 0$ wird dann

$$K(k) \to +\frac{\pi}{2}; \quad E(k) \to +\frac{\pi}{2},$$

und für $|k| < 1$ gelten die hypergeometrischen Entwicklungen

$$K(k) = \frac{\pi}{2} \cdot F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2);$$

$$E(k) = \frac{\pi}{2} \cdot F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2).$$

Setzt man in (11), oder

$$k_{n-1} = \frac{2\sqrt{k_n}}{1+k_n} \quad \ldots\ldots\ldots \quad (48)$$

$$k_n = e^{i\varphi}, \text{ mit } |\varphi| < \pi,$$

und wird $\sqrt{k_n}$ durch $+e^{\frac{i\varphi}{2}}$ bestimmt, so wird

$$k_{n-1} = \frac{2}{e^{-\frac{i\varphi}{2}} + e^{+\frac{i\varphi}{2}}} = \frac{1}{\cos\frac{\varphi}{2}}, \text{ mit } \left|\frac{\varphi}{2}\right| < \frac{\pi}{2}.$$

Während also k_n den Einheitskreis $|k_n| = 1$ durchläuft, durchläuft k_{n-1} die reelle Gerade von $+1$ nach $+\infty$.

Weiter folgt aus (48), dass

$$\begin{aligned} \text{einerseits} \quad k_n &= 0 \quad \rightarrow k_{n-1} = 0, \\ \text{andrerseits} \quad k_n &= \infty \quad \rightarrow k_{n-1} = 0, \end{aligned}$$

d.h.:

Das Innere der ganzen von $k_{n-1} = +1$ nach $+\infty$ aufgeschlitzten k_{n-1}-Ebene wird durch die Transformation (48)

einerseits auf das *Innere* des Einheitskreises $|k_n| = 1$,
andrerseits auf das *Aeussere* des Einheitskreises $|k_n| = 1$

abgebildet.

Weil wir nur die erste Alternative wünschen, sind die Wurzelzeichen in (9) oder besser in

$$k_n = \frac{1 - \sqrt{1 - k_{n-1}^2}}{1 + \sqrt{1 - k_{n-1}^2}}$$

so zu bestimmen, dass umgekehrt mit $k_{n-1} = 0$ *nur* noch $k_n = 0$ korrespondiert, d.h. es soll

$$\lim_{k_{n-1} \to 0} \arg \sqrt{1 - k_{n-1}^2} = 0 \quad \ldots\ldots\ldots \quad (49)$$

sein.

Man erreicht dieses Ergebnis durch die Feststellung

$$\left.\begin{aligned} |\arg(1 - k_{n-1})| &< \pi \\ |\arg(1 + k_{n-1})| &< \pi \end{aligned} \quad (n \geqq 2) \right\} \quad \ldots\ldots\ldots \quad (50)$$

Weil die Argumente von $1 + k_{n-1}$ und $1 - k_{n-1}$ für komplexe Werte von k_{n-1} ein entgegengesetztes Zeichen erhalten, so ist

$$\left.\begin{aligned} |\arg \sqrt{1 - k_{n-1}^2}| &= \tfrac{1}{2} |\arg(1 - k_{n-1}) + \arg(1 + k_{n-1})| \leqq \\ &\leqq \tfrac{1}{2} \operatorname{Max.} \{ |\arg(1 - k_{n-1})|, |\arg.(1 + k_{n-1})| \} < \frac{\pi}{2} \end{aligned} \right\} \quad (50')$$

Durch die Verabredung (50) bzw. (50′) sind alle vorkommenden Grössen k_n und $\sqrt{1 - k_n^2}$ eindeutig bestimmt. Jeder beliebige Punkt k_1 aus dem Innern der von $+1$ nach $+\infty$ aufgeschlitzten k_1-Ebene wird dann durch (48) (oder durch ihre Umkehrung) in einen Punkt k_2 *innerhalb* des Einheitskreises G_1 $(= |k_2| < 1)$ transformiert.

Durch iterierte Transformation wird k_2 in einen Punkt k_3 *innerhalb* des Gebietes G_2 transformiert, wo G_2 ein Teilbereich von G_1 ist, u.s.w.

Wegen (48) ist

$$|\sqrt{k_n}| \leqq \frac{1 + |k_n|}{2} |k_{n-1}| < |k_{n-1}| \quad (n \geqq 2)$$

also, wegen $|k_2| < 1$

$$|k_n| < |k_{n-1}| \quad (n \geqq 2)$$

und

$$1 > |k_2| > |k_3| \ldots . > |k_n| \ldots . \quad \ldots \ldots \quad (51)$$

In derselben Weise wird durch die Transformation

$$l_{n-1} = \frac{2\sqrt{l_n}}{1 + l_n} \quad (\text{vgl. } (33))$$

unter der Feststellung

$$\begin{aligned} |arg\,(1 - l_{n-1})| &< \pi \\ |arg\,(1 + l_{n-1})| &< \pi \end{aligned} \quad (n \geqq 2)$$

$$\left.\begin{aligned} |arg \sqrt{1 - l_{n-1}^2}| &= \tfrac{1}{2}\,|arg\,.(1 - l_{n-1}) + arg\,.(1 + l_{n-1})| \leqq \\ &\leqq \tfrac{1}{2}\, Max \{|arg\,(1 - l_{n-1})|,\ |arg\,.(1 + l_{n-1})|\} < \frac{\pi}{2} \end{aligned}\right\} . \quad (52)$$

jeder beliebige Punkt l_{n-1} aus dem Innern der von $+\,1$ nach $+\,\infty$ aufgeschlitzten l_{n-1}-Ebene in einen Punkt l_n innerhalb des Einheitskreises $|l_n| < 1$ transformiert.

Hieraus ergibt sich

$$1 > |l_2| > |l_3| \ldots . > |l_n| \ldots . \quad \ldots \ldots \quad (53)$$

Insbesondere ergibt sich aus (50′) und (52) für $n = 2$

$$|arg\,.\sqrt{1 - k_1^2}| = |arg\,.l_1| < \frac{\pi}{2};$$

$$|arg\,.\sqrt{1 - l_1^2}| = |arg\,.k_1| < \frac{\pi}{2}.$$

Die letztere Ungleichung bedeutet keinerlei Beschränkung in der Wahl der Grösse k_1, denn für die Integrale $K(k)$ und $E(k)$ ist nicht die Wahl der Grösse k, sondern nur die der Grösse k^2 wesentlich.

Definiert man also

$$|arg\ k^2| < \pi$$

so ist, wie oben

$$|arg\,.\,k| < \frac{\pi}{2}.$$

Die Integrale $K(k)$ und $E(k)$ sind dann analytische Funktionen von k^2 im Innern der von $k^2 = +1$ nach $+\infty$ aufgeschlitzten k^2-Ebene.

Weil die Entwicklungen K, K^*, E und E^* für reelles k mit $0 \leqq k < 1$ gelten, so findet man durch analytische Fortsetzung, wenn man weiter in (35) für $log\,\frac{4}{l_q}$ den Hauptwert wählt, zusammenfassend:

Wenn

$$|arg\,(1 \pm k_{n-1})| < \pi;\ |arg\,(1 \pm (l_{n-1})| < \pi;$$

also

$$\left.\begin{aligned} &|arg\,(1 \pm k_{n-1})| < \pi;\ |arg\,(1 \pm (l_{n-1})| < \pi; \\ &|arg\,.\sqrt{1 - k_{n-1}^2}| < \frac{\pi}{2};\ |arg\,.\sqrt{1 - l_{n-1}^2}| < \frac{\pi}{2}, \end{aligned}\right\} \quad (n \geqq 2)$$

so gelten alle Entwicklungen K, K^*, E, sowie die aus (47) *hervorgehenden Entwicklungen* E^* *von II an für* **alle** *komplexen Werte von* k *in der von* $k = +1$ *nach* $+\infty$ *aufgeschlitzten rechten Halbebene* $\Re(k) \geqq 0$ (oder für alle komplexen Werte $z = k^2$ in der von $z = +1$ nach $+\infty$ aufgeschlitzten z-Ebene).

Die $n$ten Entwicklungen stellen dann innerhalb des Einheitskreises $|k_n| = 1$, bzw. $|l_n| = 1$ konvergente Potenzreihen dar (ausgenommen die Reihen KK (vgl. Proc. Ned. Akad. v. Wetensch., XLIV, N⁰. 8, 1941, S. 973)).

Schliesslich ist zu bemerken, dass man durch geeignete Wahl der Reihen K oder K^*, (bzw. E oder E^*) nach einigen Transformationen immer für *alle* komplexen Werte von k mit $\Re(k) \geqq 0$ eine starke Konvergenz erzeugen kann.

Wegen (51) ist

$$|k_n| < 1 \quad \text{für} \quad n > 1,$$

und wegen (53)

$$|l_n| < 1 \quad \text{für} \quad n > 1.$$

Wenn

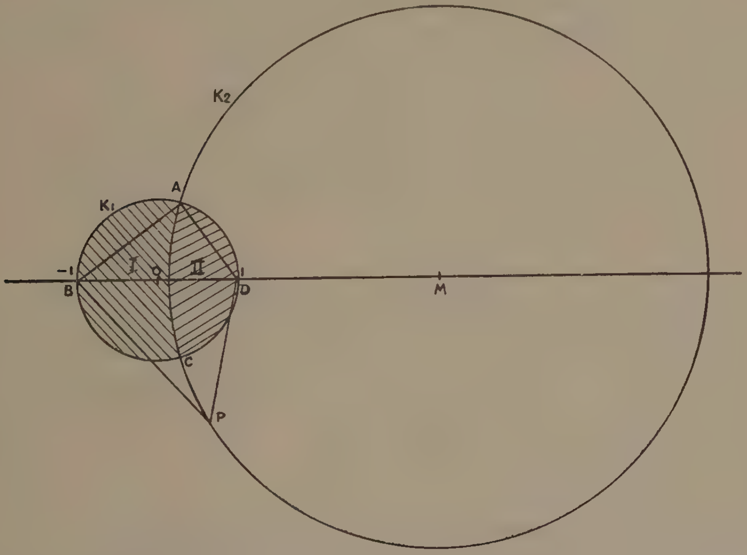
$$|k_3| \leqq \varrho < 1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (54)$$

ist, so ist, wegen (48)

$$|k_n| \leqq \left(\frac{1+|k_n|}{2}\right)^2 |k_{n-1}|^2 < |k_{n-1}|^2 \quad \text{für} \quad n \geqq 3,$$

also allgemein

$$|k_n| \leqq |k_3|^{2^{n-3}} \leqq \varrho^{2^{n-3}} \quad \text{für} \quad n \geqq 3.$$



K_1 ist der Einheitskreis um O, K_2 ist der Apollonische Kreis bei dem Verhältnis $\dfrac{PB}{PD} = \frac{4}{3}$, also $AB = 1{,}6$; $AD = 1{,}2$.

k_3 liegt innerhalb des Kreises K_1.

1. Wenn k_3 im Gebiet $\mathrm{I} \equiv ABC$ (zwischen K_1 und K_2, K_2 eingeschlossen) liegt, so ist

$$|1 + k_3| < AB = 1{,}6$$

also, wegen (48)

$$|k_3| = \left|\frac{1+k_3}{2}\right|^2 |k_2|^2 < 0{,}8^2$$

Man kann also in (54) $\varrho = 0{,}8^2$ wählen, und

$$|k_n| \leqslant (0{,}8)^{2^{n-2}}, \quad n \geqslant 3.$$

Die Reihen (12) und (42) liefern dann für $n \geqslant 3$ stark konvergente Entwicklungen.

2. Wenn k_3 im Gebiet II $\equiv ACD$ (zwischen K_1 und K_2, K_2 eingeschlossen) liegt, so ist

$$\left|\frac{1-k_3}{1+k_3}\right| \leqslant \tfrac{3}{4}$$

also, wegen (10)

$$|\sqrt{1-k_2^2}| \leqslant \tfrac{3}{4}.$$

Setzt man in (32)

$$l_1 = \sqrt{1-k_2^2}, \quad \text{also} \quad |l_1| \leqslant \tfrac{3}{4},$$

so folgt aus

$$l_{n-1} = \frac{2\sqrt{l_n}}{1+l_n}$$

wie oben,

$$|l_n| \leqslant |l_1|^{2^{n-1}} \leqslant (\tfrac{3}{4})^{2^{n-1}},$$

und die Reihen (35) und (47) konvergieren schon stark für $n \geqslant 1$, *wenn man wenigstens* $k_2 = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}$ *statt* k_1 *verwendet.*

Wenn $|k|$ gross ist, kann man auch mit Vorteil die Transformationen

$$k \,.\, K(k) = K\left(\frac{1}{k}\right) + i\,\{\text{sgn.}\, I(k^2)\}\, K\left(\sqrt{1-\frac{1}{k^2}}\right)$$

und

$$k \,.\, E(k) = (1-k^2)\, K\left(\frac{1}{k}\right) + k^2 E\left(\frac{1}{k}\right) + i\,\{\text{sgn.}\, I(k^2)\} \left\{ K\left(\sqrt{1-\frac{1}{k^2}}\right) - k^2 E\left(\sqrt{1-\frac{1}{k^2}}\right) \right\}$$

anwenden, wo $K\left(\sqrt{1-\frac{1}{k^2}}\right)$ durch die Entwicklungen K^*, und $E\left(\sqrt{1-\frac{1}{k^2}}\right)$ durch die Entwicklungen E^* berechnet wird $\left(l_q = \frac{1}{k}\right)$ 1).

Dordrecht, 16. Dezember 1941.

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1) Der Beweis wird in der nächsten Nr. der Proceedings erscheinen.

Mathematics. — *Ueber die Entwicklung der unvollständigen elliptischen Integrale erster und zweiter Art in stark konvergente Reihen.* IV. Von S. C. VAN VEEN. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of December 27, 1941.)

§ 3. *α und β in der Nähe von $\frac{\pi}{2}$.*

Im folgenden wird zur Abkürzung gesetzt

$$\left.\begin{aligned} &\triangle = \sqrt{1-\sin^2\alpha \,.\, \sin^2\beta} = \sqrt{\cos^2\alpha + \cos^2\beta\,\sin^2\alpha}\,; \\ &x = \frac{1-\sin\alpha}{\sqrt{\sin\alpha}}\,; \qquad\qquad y = \frac{\triangle + \cos\beta\sqrt{\sin\alpha}}{\triangle - \cos\beta\sqrt{\sin\alpha}}\,; \end{aligned}\right\} \quad . \quad . \quad (36)$$

$$\left.\begin{aligned} F(\sin\alpha,\beta) &= \int_0^{\beta}\frac{d\varphi}{\sqrt{1-\sin^2\alpha\,.\,\sin^2\varphi}} = \int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-\sin^2\alpha\,.\,\sin^2\varphi}} - \\ &- \int_{\beta}^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\cos^2\alpha+\cos^2\varphi-\cos^2\alpha\cos^2\varphi}} = K(\alpha) - \int_0^{\frac{\pi}{2}-\beta}\frac{d\varphi}{\sqrt{\cos^2\alpha+\sin^2\varphi-\cos^2\alpha\sin^2\varphi}}\,. \end{aligned}\right\} \quad (37)$$

Setzt man in das letzte Integral

$$\sin\varphi = \operatorname{cotg}\alpha\,.\,\operatorname{sh} u,$$

so findet man

$$\int_0^{\frac{\pi}{2}-\beta}\frac{d\varphi}{\sqrt{\cos^2\alpha+\sin^2\varphi-\cos^2\alpha\sin^2\varphi}} = \frac{1}{\sin\alpha}\int_0^{\log\left(\frac{\cos\beta\sin\alpha+\triangle}{\cos\alpha}\right)}\frac{du}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2 u}}\,. \qquad (38)$$

Ebenso ist

$$\left.\begin{aligned} E(\sin\alpha,\beta) &= \int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2\alpha\sin^2\varphi}\,d\varphi - \int_0^{\frac{\pi}{2}-\beta}\sqrt{\cos^2\alpha+\sin^2\varphi-\cos^2\alpha\,.\,\sin^2\varphi}\,d\varphi = \\ &= E(\alpha) - \frac{\cos^2\alpha}{\sin\alpha}\int_0^{\log\left(\frac{\cos\beta\sin\alpha+\triangle}{\cos\alpha}\right)}\frac{\operatorname{ch}^2 u\,du}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2 u}}\,. \end{aligned}\right\} \quad (39)$$

In (38) ist

$$\left.\begin{aligned}\int_0^{\log\left(\frac{\cos\beta\sin\alpha+\Delta}{\cos\alpha}\right)} \frac{du}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,sh^2u}} &= \int_0^{\log\left(\frac{\cos\beta\sin\alpha+\Delta}{\cos\alpha}\right)} \frac{du}{\sqrt{\left(ch^2u-\frac{sh^2u}{\sin\alpha}\right)^2-\frac{sh^2u\,.\,ch^2u\,(1-\sin\alpha)^2}{\sin^2\alpha}}} = \\ &= \sin\alpha \int_0^{\log\left(\frac{\cos\beta.\sin\alpha+\Delta}{\cos\alpha}\right)} \frac{du}{ch^2u\,(\sin\alpha-th^2u)\sqrt{1-\frac{th^2u\,(1-\sin\alpha)^2}{(\sin\alpha-th^2u)^2}}}.\end{aligned}\right\} \text{(40}$$

Setzt man hier

$$th\,u=\sqrt{\sin\alpha}\,.\,th\,v;\; y=\frac{\Delta+\cos\beta\sqrt{\sin\alpha}}{\Delta-\cos\beta\sqrt{\sin\alpha}};\quad .\;\;.\;\;.\;\;.\quad (41)$$

so ist

$$\left.\begin{aligned}\int_0^{\log\left(\frac{\cos\beta\sin\alpha+\Delta}{\cos\alpha}\right)} \frac{du}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,sh^2u}} &= \sqrt{\sin\alpha}\int_0^{\frac{\log y}{2}} \frac{dv}{ch^2v\,.\,(1-th^2v)\sqrt{1-\frac{(1-\sin\alpha)^2\,th^2v}{\sin\alpha\,.\,(1-th^2v)^2}}} = \\ = \sqrt{\sin\alpha}\int_0^{\frac{\log y}{2}} \frac{dv}{\sqrt{1-x^2\,sh^2v\,.\,ch^2v}} &= \sqrt{\sin\alpha}\,.\sum_{n=0}^{\infty}\left(\frac{1.3.5...(2n-1)}{2.4.6...2n}\right)x^{2n}\int_0^{\frac{\log y}{2}} sh^{2n}v\,.\,ch^{2n}v\,.\,dv = \\ = \sqrt{\sin\alpha}\,.\sum_{n=0}^{\infty}\left(\frac{1.3.5...(2n-1)}{2.4.6...2n}\right)x^{2n}\int_0^{\frac{\log y}{2}}&\left(\frac{e^{2v}-e^{-2v}}{4}\right)^{2n} dv.\end{aligned}\right\} \text{(4}$$

Nach (26) III ist

$$\int_0^{\frac{\log y}{2}}\left(\frac{e^{2v}-e^{-2v}}{4}\right)^{2n} dv=\frac{(-1)^n}{2^{2n+1}}\left(\frac{1.3.5...(2n-1)}{2.4.6...2n}\right)\log y+\frac{y^{2n}}{4^{2n+1}}\sum_{\substack{p=0\\ p\neq n}}^{2n}(-1)^p\binom{2n}{p}\frac{y^{-2p}}{n-p}.\quad (43)$$

Schliesslich ergibt sich aus (37)—(43):

$$\left.\begin{aligned}F(\sin\alpha,\beta)=K(\alpha)-\frac{1}{2\sqrt{\sin\alpha}}\log y\,.\sum_{n=0}^{\infty}(-1)^n\left(\frac{1.3.5...(2n-1)}{2.4.6...2n}\right)^2\left(\frac{x}{2}\right)^{2n} \\ -\frac{1}{4\sqrt{\sin\alpha}}\sum_{n=1}^{\infty}\left(\frac{1.3.5...(2n-1)}{2.4.6...2n}\right)\left(\frac{xy}{4}\right)^{2n}\sum_{\substack{p=0\\ p\neq n}}^{2n}(-1)^p\binom{2n}{p}\frac{y^{-2p}}{n-p}.\end{aligned}\right\}(44)$$

Bemerkt man noch, dass

$$\left.\begin{aligned} \frac{xy}{4} &= \frac{1-\sin\alpha}{4\sqrt{\sin\alpha}} \cdot \frac{\Delta+\cos\beta\sqrt{\sin\alpha}}{\Delta-\cos\beta\sqrt{\sin\alpha}} = \\ &= \frac{1-\sin\alpha}{4\sqrt{\sin\alpha}} \cdot \frac{(\Delta+\cos\beta\sqrt{\sin\alpha})^2}{(1-\sin\alpha)(1+\sin\alpha\,.\,\sin^2\beta)} = \frac{(\Delta+\cos\beta\sqrt{\sin\alpha})^2}{4\sqrt{\sin\alpha}\,.\,(1+\sin\alpha\,.\,\sin^2\beta)}, \end{aligned}\right\} \quad (45)$$

so gewinnt man die für α und β in der Nähe von $\frac{\pi}{2}$ stark konvergente Entwicklung

$$\left.\begin{aligned} F(\sin\alpha,\beta) &= K(\alpha) - \frac{1}{2\sqrt{\sin\alpha}}\,.\,\log y \sum_{n=0}^{\infty}(-1)^n\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right)^2\left(\frac{x}{2}\right)^{2n} \\ &- \frac{1}{4\sqrt{\sin\alpha}}\sum_{n=1}^{\infty}\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right)\left\{\frac{(\Delta+\cos\beta\sqrt{\sin\alpha})^2}{4\sqrt{\sin\alpha}\,.\,(1+\sin\alpha\,.\,\sin^2\beta)}\right\}^{2n}\sum_{\substack{p=0\\p\neq n}}^{2n}(-1)^p\binom{2n}{p}\frac{y^{-2p}}{n-p}. \end{aligned}\right\} \quad (46)$$

Das Hauptglied ist

$$F(\sin\alpha,\beta) \approx K(\alpha) - \frac{1}{2\sqrt{\sin\alpha}}\,.\,\log\frac{\Delta+\cos\beta\sqrt{\sin\alpha}}{\Delta-\cos\beta\sqrt{\sin\alpha}}, \quad . \quad (47)$$

wo $K(\alpha)$ durch die Formeln aus V.E. I. (II) bestimmt ist.

Wegen

$$\left|\sum_{\substack{p=0\\p\neq n}}^{2n}(-1)^p\binom{2n}{p}\frac{y^{-2p}}{n-p}\right| < \sum_{p=0}^{2n}\binom{2n}{p}y^{-2p} = (1+y^{-2})^{2n} = \left\{\frac{2(\Delta^2+\cos^2\beta\,.\,\sin\alpha)}{(\Delta+\cos\beta\sqrt{\sin\alpha})^2}\right\}^{2n}$$

ist die letzte Reihe in (46), wegen (45), stärker konvergent als die Reihe

$$\sum_{n=1}^{\infty}\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right)\left(\frac{\Delta^2+\cos^2\beta\sin\alpha}{2\sqrt{\sin\alpha}(1+\sin\alpha\,.\,\sin^2\beta)}\right)^{2n}. \quad . \quad . \quad (48)$$

Diese Reihe konvergiert für

$$\frac{\Delta^2+\cos^2\beta\,.\,\sin\alpha}{2\sqrt{\sin\alpha}\,.\,(1+\sin\alpha\,.\,\sin^2\beta)} = \frac{1+\sin\alpha}{2\sqrt{\sin\alpha}}\cdot\left(\frac{1-\sin\alpha\,.\,\sin^2\beta}{1+\sin\alpha\,.\,\sin^2\beta}\right) = \frac{\bar{x}}{2}\left(\frac{1-\sin\alpha\,.\,\sin^2\beta}{1+\sin\alpha\,.\,\sin^2\beta}\right) < 1$$

mit

$$\bar{x} = \frac{1+\sin\alpha}{\sqrt{\sin\alpha}}, \qquad (\geqq 2).$$

Hieraus ergibt sich als *hinreichende* Bedingung für die Konvergenz der zweiten Reihe aus (46)

$$1 \geqq \sin^2\beta > \frac{1}{\sin\alpha}\left(\frac{\bar{x}-2}{\bar{x}+2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

(vgl. (28)). Die erste Reihe konvergiert dann auch, weil

$$\sin\alpha > \frac{\bar{x}-2}{\bar{x}+2} = \left(\frac{1-\sqrt{\sin\alpha}}{1+\sqrt{\sin\alpha}}\right)^2$$

oder

$$\frac{x}{2}=\frac{1-\sqrt{\sin\alpha}}{2\sqrt{\sin\alpha}}<\frac{1+\sqrt{\sin\alpha}}{2}<1$$

ist, *und* (49) *bildet somit eine hinreichende Bedingung für die Konvergenz der Reihen* (46).

Zur Entwicklung des elliptischen Integrales zweiter Art setzen wir

$$\log\left(\frac{\cos\beta\,.\,\sin\alpha+\Delta}{\cos\alpha}\right)=\gamma,$$

und betrachten wir zuerst

$$\int\limits_0^{\gamma}\frac{\operatorname{sh}^2u\,du}{(1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u)^{\frac{3}{2}}}=\int\limits_0^{\gamma}\frac{\operatorname{ch}^2u\,.\,\operatorname{sh}^2u\,.\,du}{(1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u)^{\frac{3}{2}}}-\int\limits_0^{\gamma}\frac{\operatorname{sh}^4u\,.\,du}{(1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u)^{\frac{3}{2}}}=$$

$$=\frac{1}{\operatorname{cotg}^2\alpha}\left[\frac{\operatorname{ch}u\,.\,\operatorname{sh}u}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u}}\right]_0^{\gamma}-\frac{1}{\operatorname{cotg}^2\alpha}\int\limits_0^{\gamma}\frac{\operatorname{ch}^2u+\operatorname{sh}^2u}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u}}\,du-$$

$$-\operatorname{tg}^2\alpha\int\limits_0^{\gamma}\frac{\operatorname{sh}^2u\,du}{(1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u)^{\frac{3}{2}}}+\operatorname{tg}^2\alpha\int\limits_0^{\gamma}\frac{\operatorname{sh}^2u\,du}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u}},$$

oder

$$\frac{1}{\cos^2a}\int\limits_0^{\gamma}\frac{\operatorname{sh}^2u\,du}{(1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u)^{\frac{3}{2}}}=\frac{\sin^3\alpha\,.\,\cos\beta\,.\,\Delta}{\cos^4\alpha\,.\,\sin\beta}-\operatorname{tg}^2\alpha\int\limits_0^{\gamma}\frac{\operatorname{ch}^2u\,.\,du}{\sqrt{1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u}}.\tag{50}$$

Aus (50) und (39) ergibt sich

$$E(\sin\alpha,\beta)=E(\alpha)-\operatorname{cotg}\beta\,.\,\Delta+\frac{\cos^2\alpha}{\sin^3\alpha}\int\limits_0^{\gamma}\frac{\operatorname{sh}^2u\,du}{(1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u)^{\frac{3}{2}}}.\tag{51}$$

Wie oben findet man (vgl. (40) und (42))

$$\left.\begin{aligned}&\int\limits_0^{\gamma}\frac{\operatorname{sh}^2u\,du}{(1-\operatorname{cotg}^2\alpha\,.\,\operatorname{sh}^2u)^{\frac{3}{2}}}=\sin^3\alpha\int\limits_0^{\gamma}\frac{\operatorname{sh}^2u\,.\,du}{\operatorname{ch}^6u\,.\,(\sin\alpha-\operatorname{th}^2u)^3\left(1-\dfrac{\operatorname{th}^2u\,(1-\sin\alpha)^2}{(\sin\alpha-\operatorname{th}^2u)^2}\right)^{\frac{3}{2}}}=\\&=(\sin\alpha)^{\frac{3}{2}}\int\limits_0^{\frac{\log y}{2}}\frac{\operatorname{th}^2v\,(1-\sin\alpha\,.\,\operatorname{th}^2v)\,dv}{\operatorname{ch}^2v\,.\,(1-\operatorname{th}^2v)^3\left(1-\dfrac{(1-\sin\alpha)^2\operatorname{th}^2v}{\sin\alpha\,(1-\operatorname{th}^2v)^2}\right)^{\frac{3}{2}}}=(\sin\alpha)^{\frac{3}{2}}\int\limits_0^{\frac{\log y}{2}}\frac{\operatorname{sh}^2v\,(\operatorname{ch}^2v-\sin\alpha\,.\,\operatorname{sh}^2v)\,dv}{(1-x^2\operatorname{sh}^2v\,.\,\operatorname{ch}^2v)^{\frac{3}{2}}}=\\&=(\sin\alpha)^{\frac{3}{2}}\sum_{n=0}^{\infty}\left(\frac{3.5\ldots(2n+1)}{2.4\ldots2n}\right)x^{2n}\int\limits_0^{\frac{\log y}{2}}\operatorname{sh}^{2n+2}v\,.\operatorname{ch}^{2n}v\,(\operatorname{ch}^2v-\sin\alpha\,.\,\operatorname{sh}^2v)\,dv.\end{aligned}\right\}\;(52$$

Weiter ist

$$\left.\begin{aligned}
&\int_0^{\frac{\log y}{2}} sh^{2n+2}v\,.\,ch^{2n}v\,.\,(ch^2v - \sin\alpha\,.\,sh^2v)\,dv = \int_0^{\frac{\log y}{2}} sh^{2n+2}v\,.\,ch^{2n}v\,\{(1-\sin\alpha)\,ch^2v + \sin\alpha\}\,dv = \\
&= (1-\sin\alpha)\int_0^{\frac{\log y}{2}}\left(\frac{e^{2v}-e^{-2v}}{4}\right)^{2n+2}dv + \sin\alpha\int_0^{\frac{\log y}{2}}\left(\frac{e^{v}-e^{-v}}{2}\right)^{2}\left(\frac{e^{2v}-e^{-2v}}{4}\right)^{2n}dv = \\
&= (1-\sin\alpha)\int_0^{\frac{\log y}{2}}\left(\frac{e^{2v}-e^{-2v}}{4}\right)^{2n+2}dv - \frac{\sin\alpha}{2}\int_0^{\frac{\log y}{2}}\left(\frac{e^{2v}-e^{-2v}}{4}\right)^{2n}dv + \\
&\qquad\qquad + \sin\alpha\int_0^{\frac{\log y}{2}}\left(\frac{e^{2v}+e^{-2v}}{4}\right)\left(\frac{e^{2v}-e^{-2v}}{4}\right)^{2n}dv,
\end{aligned}\right\}\quad(53)$$

also, wegen (43)

$$\left.\begin{aligned}
&sh^{2n+2}v\,.\,ch^{2n}v\,(ch^2v - \sin\alpha\,.\,sh^2v)\,dv = \\
&1-\sin\alpha)\left\{\frac{(-1)^{n+1}}{2^{2n+3}}\left(\frac{1.3.5\ldots(2n+1)}{2.4.6\ldots(2n+2)}\right)\log y + \frac{y^{2n+2}}{4^{2n+3}}\sum_{\substack{p=0\\p\neq n+1}}^{2n+2}(-1)^p\binom{2n+2}{p}\frac{y^{-2p}}{n+1-p}\right\} \\
&\frac{in\,\alpha}{2}\left\{\frac{(-1)^n}{2^{2n+1}}\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right)\log y + \frac{y^{2n}}{4^{2n+1}}\sum_{\substack{p=0\\p\neq n}}^{2n}(-1)^p\binom{2n}{p}\frac{y^{-2p}}{n-p}\right\} + \frac{\sin\alpha}{2^{4n+3}}\,.\,\frac{(y-y^{-1})^{2n+1}}{2n+1}\,.
\end{aligned}\right\}\quad(54)$$

Aus (51), (52) und (54) ergibt sich endlich

$$\begin{aligned}
&(\sin\alpha,\beta) = E(\alpha) - \operatorname{cotg}\beta\,.\,\triangle + \frac{\cos^2\alpha}{(\sin\alpha)^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{1-\sin\alpha}{8}(-1)^{n+1}\left(\frac{1.3.5\ldots(2n+1)}{2.4.6\ldots(2n+2)}\right)^2(2n+2)\,.\left(\frac{x}{2}\right)^{2n}\log y \\
&\frac{\cos^2\alpha}{4\sqrt{\sin\alpha}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right)^2(2n+1)\,.\left(\frac{x}{2}\right)^{2n}\log y \\
&\frac{\cos^2\alpha}{(\sin\alpha)^{\frac{3}{2}}}\,.\,\frac{1-\sin\alpha}{4}\sum_{n=0}^{\infty}\left(\frac{3.5\ldots(2n+1)}{2.4\ldots2n}\right)x^{2n}\frac{y^{2n+2}}{4^{2n+2}}\sum_{\substack{p=0\\p\neq n+1}}^{2n+2}(-1)^p\binom{2n+2}{p}\frac{y^{-2p}}{n+1-p} \\
&\frac{\cos^2\alpha}{8\sqrt{\sin\alpha}}\sum_{n=0}^{\infty}\left(\frac{3.5\ldots(2n+1)}{2.4\ldots2n}\right)x^{2n}\frac{y^{2n}}{4^{2n}}\sum_{\substack{p=0\\p\neq n}}^{2n}(-1)^p\binom{2n}{p}\frac{y^{-2p}}{n-p} \\
&\frac{\cos^2\alpha}{2\sqrt{\sin\alpha}}\sum_{n=0}^{\infty}\left(\frac{3.5\ldots(2n+1)}{2.4\ldots2n}\right)\frac{x^{2n}}{2^{4n+2}}\,\frac{(y-y^{-1})^{2n+1}}{2n+1}\,.
\end{aligned}$$

Die letztere Reihe dieser Entwicklung geht über in

$$\frac{\cos^2\alpha}{8\sqrt{\sin\alpha}}(y-y^{-1})\sum_{n=0}^{\infty}\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right)\left(\frac{x(y-y^{-1})}{4}\right)^2=$$

$$=\frac{\cos^2\alpha}{2\sqrt{\sin\alpha}}\cdot\frac{\Delta.\cos\beta.\sqrt{\sin\alpha}}{(1-\sin\alpha)(1+\sin\alpha\sin^2\beta)}\sum_{n=0}^{\infty}\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\right)\left(\frac{1-\sin\alpha}{\sqrt{\sin\alpha}}\cdot\frac{\Delta.\cos\beta.\sqrt{\sin\alpha}}{(1-\sin\alpha)(1+\sin\alpha\sin^2\beta)}\right)^2=$$

$$=\frac{(1+\sin\alpha).\cos\beta.\Delta}{2(1+\sin\alpha.\sin^2\beta)}\frac{1}{\sqrt{1-\left(\frac{\Delta.\cos\beta}{1+\sin\alpha.\sin^2\beta}\right)^2}}=\frac{(1+\sin\alpha).\cos\beta.\Delta}{2}\cdot\frac{1}{\sin\beta.(1+\sin\alpha)}=\frac{\operatorname{cotg}\beta.\Delta}{2}$$

Man findet (mit Hervorhebung der Hauptglieder)

$$\left.\begin{aligned}
E(\sin\alpha,\beta)&=E(\alpha)-\frac{\operatorname{cotg}\beta.\Delta}{2}-\frac{\cos^2\alpha}{4\sqrt{\sin\alpha}}\log y\\
&+\frac{1+\sin\alpha}{4\sqrt{\sin\alpha}}\log y.\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{1.3.5\ldots(2n+1)}{2.4.6\ldots2n}\right)^2\left(\frac{x}{2}\right)^{2n+2}.\{(2n+1)+(2n+3)\sin\alpha\}\\
&+\frac{1+\sin\alpha}{4\sqrt{\sin\alpha}}\sum_{n=0}^{\infty}\left(\frac{3.5\ldots(2n+1)}{2.4\ldots2n}\right)\left(\frac{x.y}{4}\right)^{2n+2}\sum_{\substack{p=0\\p\neq n+1}}^{2n+2}(-1)^p\binom{2n+2}{p}\frac{y^{-2p}}{n+1-p}\\
&-\frac{\cos^2\alpha}{8\sqrt{\sin\alpha}}\sum_{n=0}^{\infty}\left(\frac{3.5\ldots(2n+1)}{2.4\ldots2n}\right)\left(\frac{x.y}{4}\right)^{2n}\sum_{\substack{p=0\\p\neq n}}^{2n}(-1)^p\binom{2n}{p}\frac{y^{-2p}}{n-p}
\end{aligned}\right\}\quad(55)$$

mit

$$x=\frac{1-\sin\alpha}{\sqrt{\sin\alpha}};\quad y=\frac{\Delta+\cos\beta\sqrt{\sin\alpha}}{\Delta-\cos\beta\sqrt{\sin\alpha}};\quad xy=\frac{(\Delta+\cos\beta\sqrt{\sin\alpha})^2}{4\sqrt{\sin\alpha}(1+\sin\alpha.\sin^2\beta)}.$$

Das Hauptglied ist (mit Vernachlässigung von $x^2\log y$, x^2y^2, u.s.w.) [1])

$$E(\sin\alpha,\beta)\approx E(\alpha)-\frac{\operatorname{cotg}\beta.\Delta}{2}-\frac{\cos^2\alpha}{4\sqrt{\sin\alpha}}\log\frac{\Delta+\cos\beta\sqrt{\sin\alpha}}{\Delta-\cos\beta\sqrt{\sin\alpha}},\quad(56)$$

wo $E(\alpha)$ durch die Formeln aus V.E. I (III) bestimmt ist.

Die Konvergenzbedingungen sind ganz dieselben wie in (46); die *hinreichende* Konvergenzbedingung wird somit durch (49) gegeben.

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[1]) Dieses Hauptglied ist in der Einleitung (Proc. XLIV, N⁰. 8, S. 976) fehlerhaft angegeben worden.

Mathematics. — *Sur quelques inégalités de la théorie des fonctions et leurs généralisations spatiales.* I. Par A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of December 27, 1941.)

§ 1. *Introduction.*

Soit

$$w = f(z) = a_1 z + a_2 z^2 + \dots \quad . \quad . \quad . \quad . \quad . \quad (1)$$

régulier et „schlicht" pour $|z| < 1$; le cercle unité est représenté sur un domaine „schlicht" et simplement connexe Ω. On a alors

$$\frac{z}{(1+|z|)^2}|a_1| \leqq |f(z)| \leqq \frac{|z|}{(1-|z|)^2}|a_1| . \quad . \quad . \quad . \quad . \quad (2)$$

Pour autant que je sais, on n'a jamais étudié les conséquences de ces inégalités pour la fonction inverse $z = F(w)$ qui représente Ω sur le cercle unité $|z| < 1$; peut-être la raison en est que $F(w)$ ne peut pas être développé dans tout Ω en une série de puissances de z. Cependant on obtient une propriété très simple.

Supposons que $z = 0$ est transformé en P_0 de Ω. Soit $G(P, P_0)$ la fonction de GREEN de Ω avec pôle P_0. On a

$$G(P, P_0) = \log \frac{1}{r_{PP_0}} - g(P, P_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

où $g(P, P_0)$ désigne la fonction harmonique dans Ω, correspondant à des valeurs-frontière $\log \frac{1}{r_{P_0 Q}}$ (Q désigne un point de la frontière Σ de Ω). On sait alors qu'on a

$$|z| = |F(w)| = e^{-G(P, P_0)},$$

et

$$F(w) = w e^{g + i\bar{g}}$$

$\bar{g}$ étant la fonction conjuguée de g. On en tire

$$|a_1| = e^{-g(P_0)}$$

où l'on a posé $g(P_0, P_0) = g(P_0)$. Il s'en suit alors de (2) que les points $P(w)$ de Ω pour lesquels $|z| = e^{-g} = r$ (constante) se trouvent tous dans l'anneau formé par les cercles de rayon $\frac{r}{(1+r)^2} e^{-g(P_0)}$ et $\frac{r}{(1-r)^2} e^{-g(P_0)}$ et de centre P_0. Autrement dit, la ligne de niveau $G = C$ se trouve entre les deux cercles de centre P_0 et de rayon respectivement

$$\frac{e^{-C}}{(1+e^{-C})^2} e^{-g(P_0)} \quad \text{et} \quad \frac{e^{-C}}{(1-e^{-C})^2} e^{-g(P_0)}.$$

En particulier, en prenant $C = 0$, on voit que le cercle de rayon $\frac{1}{4} e^{-g(P_0)}$ se trouve tout entier dans Ω. Donc, si d désigne la plus petite distance de P_0 à Σ

$$\tfrac{1}{4} e^{-g(P_0)} \leqq d$$

ou

$$g(P_0) \geqq \log \frac{1}{4d} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Si l'on pose $a_1 = 1$, donc $g(P_0) = 0$, on obtient $d \geqq \frac{1}{4}$; c'est le théorème de KOEBE.

Les inégalités (2) et le théorème de KOEBE sont donc transformées en une propriété qui n'a aucun rapport direct avec la théorie des fonctions de variable complexe: on a obtenu un théorème de la théorie du potentiel. La question se pose alors immédiatement si ce théorème peut être généralisé pour un espace à trois dimensions. Par exemple, on peut s'attendre à ce qu'on a dans ce cas

$$g(P_0) \geqq \frac{C}{d} \quad \ldots\ldots\ldots\ldots \quad (5)$$

où C désigne une constante > 0.

Remarquons que dans le cas trois-dimensionnel il peut arriver que les fonctions $G(P, P_0)$ et $g(P, P_0)$ n'existent pas au sens classique. Il faut alors substituer la solution généralisée du problème de DIRICHLET, qui par exemple peut être construite par le procédé de WIENER.

Les considérations suivantes se rapportent pour la plus grande partie à l'inégalité (4) et la possibilité d'une inégalité (5). On verra que (5) n'est possible que si l'on prend $C = 0$, sauf dans quelques cas particuliers, d'ailleurs intéressants par la relation avec une autre inégalité de la théorie des fonctions.

D'abord nous dérivons quelques inégalités auxiliaires.

§ 2. *Inégalités auxiliaires.*

1. Les inégalités que nous dérivons ici se rapportent à la mesure harmonique. Nous nous plaçons dans le cas d'un espace euclidien à trois dimensions, augmenté d'un point à l'infini P_∞.

A. Soient Ω un domaine „schlicht" et simplement connexe qui ne contient P_∞ pas comme point intérieur et V un plan. Mettons l'axe des X perpendiculaire à V et soit x l'abscisse du point d'intersection de V avec cette axe. Soit $\theta(x)$ la mesure de la partie θ_x de V qui se trouve dans $\Omega + \Sigma$. Il s'agit de trouver une borne supérieure de la mesure harmonique $\mu^{P_0}(\theta_x, \Omega_x)$ de l'ensemble θ_x relativement à la partie Ω_x de Ω à gauche de V et mesuré au point P_0 de Ω_x. Soit d la distance de P_0 à V.

Remplaçons d'abord Ω_x par la partie de l'espace à gauche de V; μ^{P_0} augmente par cela. La mesure harmonique vaut alors l'angle solide sous lequel on voit θ_x de P_0, divisé par 2π. Cet angle est maximum quand θ_x est la base d'un cône circulaire droite de sommet P_0 (la surface de la base vaut $\theta(x)$). On trouve alors par un calcul élémentaire

$$\mu^{P_0}(\theta_x, \Omega_x) \leqq 1 - \frac{d}{\sqrt{d^2 + \frac{\theta(x)}{\pi}}} \quad \ldots\ldots\ldots \quad (6)$$

B. Soit V' un second plan correspondant à l'abscisse $x' < x$ et soit P_0 dans $\Omega_{x'}$. Alors on a, comme on voit en appliquant (6) et le principe de maximum des fonctions harmoniques,

$$\mu^{P_0}(\theta_x, \Omega_x) \leqq \left(1 - \frac{x - x'}{\sqrt{(x-x')^2 + \frac{\theta(x)}{\pi}}}\right) \mu^{P_0}(\theta_{x'}, \Omega_{x'}).$$

Donc

$$\mu^{P_0}(\theta_x, \Omega_x) - \mu^{P_0}(\theta_{x'}, \Omega_{x'}) \leqq \frac{x - x'}{\sqrt{(x-x')^2 + \frac{\theta(x)}{\pi}}} \mu^{P_0}(\theta_{x'}, \Omega_{x'}).$$

$\mu^{P_0}(\theta_x, \Omega_x)$ est donc une fonction décroissante de x, de sorte que sa dérivé existe presque partout. En divisant par $x - x'$ et en faisant alors $x' \to x$, on trouve

$$\frac{d\,\mu^{P_0}(\theta_x, \Omega_x)}{dx} \leqq - \sqrt{\frac{\pi}{\theta(x)}} \cdot \mu^{P_0}(\theta_x, \Omega_x).$$

Aux points où la dérivé n'existe pas on prend la plus grande dérivé à gauche. En intégrant on trouve si $x_1 < x_2$

$$\mu^{P_0}(\theta_{x_2}, \Omega_{x_2}) \leqq \mu^{P_0}(\theta_{x_1}, \Omega_{x_1})\, e^{-\sqrt{\pi}\int\limits_{x_1}^{x_2}\frac{dx}{\sqrt{\theta(x)}}} \quad \ldots\ldots \quad (7)$$

$$(P_0 \text{ dans } \Omega_{x_1})$$

Dans le cas deux-dimensionnel ces formules deviennent respectivement

$$\mu^{P_0}(\theta_x, \Omega_x) \leqq \frac{2}{\pi} \operatorname{arc\,tg} \frac{\theta(x)}{2d} \quad \ldots\ldots\ldots \quad (6')$$

et

$$\mu^{P_0}(\theta_{x_2}, \Omega_{x_2}) \leqq \mu^{P_0}(\theta_{x_1}, \Omega_{x_1})\, e^{-\pi\int\limits_{x_1}^{x_2}\frac{dx}{\theta(x)}} \quad \ldots\ldots \quad (7')$$

Les formules (6′) et (7′) sont dûs à M. CARLEMAN[1]).

2. De la formule (7′) on peut déduire une propriété importante de la mesure harmonique. Cette propriété n'est vraie, comme on le verra, que dans le cas deux-dimensionnel, de sorte que des méthodes purement deux-dimensionnel sont indispensables pour la démonstration.

Appliquons la transformation

$$z = \log w \quad \ldots\ldots\ldots\ldots \quad (8)$$

supposé que Ω se trouve dans le plan $-z$. Le domaine Ω est représenté sur le domaine Ω' qui ne contient $w = 0$ pas comme point intérieur. Les droites $\Re z = x$ sont transformés dans les cercles $|w| = R = e^x$. Si P_0 est transformé en P'_0 on trouve, la mesure harmonique étant invariante,

$$\mu^{P_0'}(\theta'_{R_2}, \Omega'_{R_2}) \leqq \mu^{P_0'}(\theta'_{R_1}, \Omega'_{R_1})\, e^{-\pi\int\limits_{R_1}^{R_2}\frac{dR}{R\,\theta(R)}} \qquad (R_2 > R_1)$$

où désignent: θ'_R la partie du cercle $|w| = R$ intérieure à Ω', Ω'_R la partie du domaine $|w| < R$ intérieur à Ω', $R\,\theta(R)$ la mesure de θ'_R. Le point P'_0 se trouve dans Ω'_{R_1}. En remarquant que $\theta(R) \leqq 2\pi$, on trouve alors

$$\mu^{P_0'}(\theta'_{R_2}, \Omega'_{R_2}) \leqq \mu^{P'_0}(\theta'_{R_1}, \Omega'_{R_1}) \left(\frac{R_1}{R_2}\right)^{\frac{1}{2}} \quad \ldots\ldots \quad (9)$$

Cette inégalité est vraie pour chaque domaine simplement connexe, ne contenant $w = 0$ pas comme point intérieur, et d'ailleurs quelque soit la position de P'_0 dans l'intersection de Ω' et $|w| \leqq R_1$. Nous omettons dans ce qui suit les accents.

En particulier on a encore puisque $\mu^{P_0} \leqq 1$,

$$\mu^{P_0}(\theta_{R_2}, \Omega_{R_2}) \leqq \left(\frac{R_1}{R_2}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots \quad (10)$$

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[1]) Voir p. ex. R. NEVANLINNA. Eindeutige analytische Funktionen (Springer, Berlin, 1936), p. 67 e.s.

Cette inégalité nous donne la clef pour la démonstration du théorème de KOEBE. Prenons R_1 constant et faisons tendre R_2 vers ∞. On voit alors que la mesure harmonique $\mu^{P_0}(\theta_{R_2}, \Omega_{R_2})$ tend vers zéro.

Ceci n'est plus vrai dans le cas analogue trois-dimensionnel. Aussi les formules (9) et (10) n'admettent des formules analogues trois-dimensionnelles que dans des cas particuliers. On voit cela par l'exemple suivant. θ_R désigne alors la partie de la sphère de rayon R, centrée en O, intérieure à Ω (O n'est pas intérieur à Ω); Ω_R désigne l'intersection de Ω avec l'intérieur de cette sphère. Il suffit de prendre pour Ω tout l'espace sauf les points d'une ligne droite s'étendant de O à P_∞, ces deux points inclus. C'est donc un domaine analogue au domaine extrémale de KOEBE („Schlitzgebiet"). Puisque la mesure harmonique, même la capacité, d'une ligne droite est zéro dans le cas trois-dimensionnel, on voit que pour ce domaine $\mu^{P_0}(\theta_R, \Omega_R) = 1$, quelque soit R et la mesure harmonique ne tend donc pas vers zéro. Remarquons que dans le cas deux-dimensionnel la mesure harmonique d'une droite est positive.

Des méthodes spécifiquement deux-dimensionnelles, telle que la transformation (8), sont donc indispensables pour arriver aux formules (9) et (10). En ceci se trouve aussi la raison de l'impossibilité d'une extension générale du théorème de KOEBE.

Le théorème suivant donne dans le cas trois-dimensionnel la relation entre l'allure de $\mu^{P_0}(\theta_R, \Omega_R)$ pour $R \to \infty$ et la mesure harmonique:

Pour que $\mu^{P_0}(\theta_R, \Omega_R)$ tend vers zèro pour $R \to \infty$, il faut et il suffit que Ω se trouve intérieur à un domaine Ω^*, ne contenant P_∞ pas comme point intérieur, dont la frontière Σ^*, passant par O, est telle que tout sous-ensemble ouvert de Σ^*, a une mesure harmonique positive relativement à Ω^* 1)).

Admettons d'abord que la condition est vérifiée. On voit que $\mu^{P_0}(\theta_R^*, \Omega_R^*)$, où θ_R^* et Ω_R^* se rapportent à Ω^*, est une fonction décroissante de R. On a $\mu^{P_0}(\theta_R^*, \Omega_R^*) = 1 - \mu^{P_0}(\Sigma_R^*\, \Omega_R^*)$, où Σ_R^* est la partie de Σ^* intérieur à la sphère de rayon R et de centre O. Maintenant, puisque P_∞ n'est pas point intérieur de Ω^*, $\mu^{P_0}(\Sigma_R^*, \Omega_R^*)$ tend vers 1 si $R \to \infty$, donc $\mu^{P_0}(\theta_R^*, \Omega_R^*) \to 0$. On achève la démonstration pour un $\Omega \subset \Omega^*$, en remarquant que la mesure harmonique croît lorsque le domaine croît, laissant invariant la partie de la frontière dont on a pris la mesure.

La nécessité d'une mesure harmonique positive résulte immédiatement de l'exemple précédent („Schlitzgebiet"). Un autre exemple montre qu'il est nécessaire que P_∞ n'est pas intérieur à Ω^*. Il suffit de prendre pour Ω^* un domaine non-borné dont la frontière est bornée et de mesure harmonique positive. Alors $\mu^{P_0}(\Sigma_R^*, \Omega_R^*)$ tend vers le potentiel d'équilibre généralisé de Σ^*, donc pas vers 1, de sorte que $\mu^{P_0}(\theta_R^*, \Omega_R^*)$ ne tend pas vers zéro.

3. Pour qu'on a l'égalité dans (9), donc cas deux-dimensionnel, il faut d'abord que l'angle $\theta(R)$ vaut 2π. Mais ceci ne suffit pas. S'il y a un domaine, donnant l'égalité, on a

$$\mu^{P_0}(\theta_{R_2}, \Omega_{R_2})\, R_2^{\frac{1}{2}} = \mu^{P_0}(\theta_{R_1}, \Omega_{R_1})\, R_1^{\frac{1}{2}} = \varphi(P_0).$$

Donc, en prenant P_0 sur l'axe reël, à un distance d de O, pour tout R

$$\mu^{P_0}(\theta_R, \Omega_R) = \frac{\varphi(d)}{\sqrt{R}}.$$

Appliquons maintenant une transformation $\bar{r} = kr$ (r = distance de P_0, k une constante). μ^{P_0} est transformé dans la mesure harmonique de $\bar{\theta}_R$, se rapportant au domaine transformé $\bar{\Omega}$. En P_0, μ^{P_0} ne change pas. Donc

$$\varphi(kd) = \sqrt{k}\,\varphi(d)$$

---

1) Selon la terminologie de VASILESCO la frontière est alors réduite, c.a.d. ne contient aucune partie impropre à porter des données pour le problème de DIRICHLET (comp. les singularités remouvables des fonctions harmoniques).

et alors

$$\varphi(d) = C\sqrt{d}.$$

C étant une constante positive. Donc, si l'égalité est possible, il faut avoir

$$\mu^{P_0}(\theta_R, \Omega_R) = C\sqrt{\frac{d}{R}} \quad \ldots\ldots\ldots \quad (11)$$

Par des raisons de symmétrie on arrive à considérer le „Schlitzgebiet" de KOEBE. On a alors

$$\mu^{P_0}(\theta_R, \Omega_R) = \frac{4}{\pi} \operatorname{arc\,tg} \sqrt{\frac{d}{R}} \quad \ldots\ldots\ldots \quad (12)$$

donc asymptotiquement (c.a.d. pour R tendant vers ∞)

$$\mu^{P_0}(\theta_R, \Omega_R) = \frac{4}{\pi} \sqrt{\frac{d}{R}} \quad \ldots\ldots\ldots \quad (13)$$

L'égalité dans (9) est donc asymptotiquement possible et l'inégalité (9) ne peut donc, au moins asymptotiquement, pas être améliorée.

§ 3. *Le théorème de* KOEBE (*deux-dimensions*).

1. Soit Ω un domaine simplement connexe, dont P_∞ n'est pas un point intérieur; soit P_0 un point intérieur à plus petite distance d de Σ; nous ne considérons dans ce paragraphe que le cas deux-dimensionnel. Il existe un point de Σ dont la distance à P_0 vaut d; nous prenons ce point comme l'origine O. Dans ce qui suit nous considérons la famille D_d de tous les domaines tel que Ω; pour tous ces domaines la distance de P_0 à la frontière vaut donc d, une constante. D'ailleurs nous pouvons arranger par une rotation convenable que le point de la frontière à distance d de P_0 est pour tous les domaines considérés le même point O. Nous démontrons alors le théorème suivant:

Il existe un nombre positif k tel que pour tout domaine appartenant à la famille D_d on a

$$g(P_0) \geqq \log \frac{k}{d}. \quad \ldots\ldots\ldots \quad (14)$$

D'abord, nous montrons l'existence d'une fonction $f(d) \not\equiv -\infty$, telle que $g(P_0) \geqq f(d)$.

Supposons une fonction $f(d)$, valable pour tous les domaines de D_d, n'existait pas. Choisissons un nombre $f_1(d)$; il existe alors un domaine Ω, tel que $g_1(P_0) < f_1(d)$.

Supposons $g_1(P_0) > f_2(d)$ où $f_2(d) < f_1(d)$. Alors f_2 n'est pas non plus valable pour tous les Ω et il existe donc un Ω_2 tel que $g_2(P_0) < f_2(d)$. En continuant ainsi, on voit qu'il existe une suite de domaines $\{\Omega_n\}$ telle que la suite correspondante des $g_n(P_0)$ tend vers $-\infty$:

$$\lim_{n\to\infty} g_n(P_0) = -\infty. \quad \ldots\ldots\ldots \quad (15)$$

Ecrivons maintenant $g_n(P_0)$ comme potentiel d'une distribution de masse positive, répartie sur la frontière Σ_n:

$$g_n(P, P_0) = \int_{\Sigma_n} \log \frac{1}{r_{PQ}} \, d\mu_n^{P_0}(e_Q), \quad \ldots\ldots \quad (16a)$$

$$g_n(P_0) = \int_{\Sigma_n} \log \frac{1}{r_{P_0 Q}} \, d\mu_n^{P_0}(e_Q), \quad \ldots\ldots \quad (16b)$$

où $\mu_n^{P_0}(e)$ désigne la mesure harmonique en P_0 du sous-ensemble e de Σ_n relativement à Ω_n.

Il suit de (16*b*) que l'existence d'une fonction $f(d) \neq -\infty$ dépend exclusivement de l'allure de $\mu_n^{P_0}$ à l'infini. En effet, si la masse de cette distribution qui se trouve extérieure à un cercle de rayon R et de centre O, tend vers zéro suffisamment vite quand $R \rightarrow \infty$, de sorte que les intégrales (16*b*) restent bornées et ne peuvent donc tendre vers $-\infty$, on arrive à une contradiction avec (15). Inversement, si cette masse ne tendait pas vers zéro, ou au moins pas suffisamment vite, il n'y aurait pas de contradiction avec (15). Il existerait alors des domaines dont $g(P_0)$ diffère arbitrairement peu de $-\infty$ et on aurait $f(d) = -\infty$.

Or, il résulte immédiatement de (10) que pour les domaines de D_d — pour ces domaines (10) est valable — la masse extérieure à un cercle de rayon R et de centre O tend vers zéro. Prenons R_1 constant et posons $R_2 = R$; soit Σ_R la partie de Σ extérieur au cercle.

Alors

$$\mu^{P_0}(\theta_R, \Omega_R) = 1 - \mu^{P_0}(\Sigma - \Sigma_R, \Omega_R) \geqq 1 - \mu^{P_0}(\Sigma - \Sigma_R, \Omega) = \mu^{P_0}(\Sigma_R, \Omega)$$

Donc

$$\mu^{P_0}(\Sigma_R, \Omega) \leqq \left(\frac{R_1}{R}\right)^{\frac{1}{2}}$$

et la masse tend vers zéro comme $R^{-\frac{1}{2}}$ pour $R \rightarrow \infty$. En vertu de (16*b*) il s'agit alors évidemment de voir que l'intégrale

$$\int_a^\infty \log \frac{1}{R} \, d R^{-\frac{1}{2}}$$

est convergente [1]). Or ceci est vraie comme on le voit par une intégration par parties. On est donc arrivé à une contradiction avec (15) et il existe donc une fonction $f(d) \neq -\infty$.

Pour arriver à (14) appliquons une homothétie de centre P_0

$$\bar{r} = p r \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

où $p > 0$, r et $\bar{r}$ distances à $P_0 \cdot g(P, P_0)$ est transformé dans une fonction harmonique qui a les valeurs de $g(P, P_0)$ aux points homothétiques; en particulier elle a les valeurs-frontière $\log \dfrac{1}{P_0 Q}$ aux points $\overline{Q}$, si Q et $\overline{Q}$ se correspondent.

Puisque $\log \dfrac{1}{\bar{r}} = \log \dfrac{1}{p} + \log \dfrac{1}{r}$ elle diffère donc de $\bar{g}(P, P_0)$ par la constante $\log \dfrac{1}{p}$.

Donc

$$\bar{g}(P_0) = g(P_0) + \log \frac{1}{p}.$$

---

[1]) Il faut remarquer ici que du fait que $\mu^{P_0} < \left(\frac{R_1}{R_2}\right)^{\frac{1}{2}}$ il ne résulte pas encore que sur tout ensemble μ^{P_0} vaut au plus la masse qui se trouve sur cet ensemble à cause de la distribution correspondant à $\left(\frac{R_1}{R_2}\right)^{\frac{1}{2}}$. En effet (10) n'exprime qu'une relation entre les masses totales extérieures à un cercle et non une relation entre les masses sur un ensemble quelconque. On peut éliminer cette difficulté par un déplacement convenable des masses de la distribution μ^{P_0} en direction de R croissant — qui fait donc diminuer le potentiel — tel qu'après ce déplacement μ^{P_0} a la propriété mentionnée. Alors il s'en suit la majoration utilisée, qui est donc vraie à fortiori avant le déplacement.

Il s'en suit

$$f(pd) = f(d) + \log \frac{1}{p}$$

donc en prenant $d = 1$ et en remplaçant alors p par d,

$$f(d) = f(1) + \log \frac{1}{d}$$

$$f(d) = \log \frac{k}{d}$$

où

$$k = e^{f(1)};\ k > 0.$$

On peut aisément donner une borne inférieure pour k. Prenons $d = 1$ et dans (10), $R_1 = 1$. Alors on a [1])

$$f(1) > -\int_1^\infty \log \frac{1}{R+1}\, d R^{-\frac{1}{2}}.$$

On trouve par un calcul élémentaire

$$f(1) = \log \tfrac{1}{2} - \frac{\pi}{2}.$$

Donc

$$k > \tfrac{1}{2}\, e^{-\frac{\pi}{2}} \quad \ldots\ldots\ldots\ldots \quad (18)$$

On sait que la borne exacte pour k est $\frac{1}{4}$ et que cette borne est accessible. Notre méthode ne conduit pas à cette borne. Il faudrait pour cela des limitations plus exactes que celle exprimée par (10); il semble problématique s'il existent de telles limitations.

2. Terminons ce paragraphe par quelques remarques concernant les formules (2) et ses conséquences relativement aux courbes $G = C\,(C > 0)$. La plus courte distance d n'est maintenant plus une constante.

Considérons d'abord les domaines simplement connexe dont P_0 est intérieur et pour lesquels $g(P_0) = 0$. Démontrons que dans ce cas il existent des fonctions positives $\varphi\,(C)$ et $\psi\,(C)$ telles que les courbes $G(P, P_0) = C$ se trouvent intérieures à l'anneau formé par les cercles de rayon $\varphi\,(C)$ et $\psi\,(C)$ et de centre P_0.

Supposons, par impossible, que $\varphi\,(C) \equiv 0$. Alors il existait une suite de domaines $\{\Omega_n\}$ telle que la plus courte distance de P_0 à la courbe $G_n = C$ tendait vers zéro si $n \to \infty$. Autrement dit: dans chaque environ de P_0 il y aurait pour $n > N$ des points P des domaines Ω_n tels que

$$G_n(P, P_0) = \log \frac{1}{r_{P, P_0}} - g_n(P, P_0) < C$$

Il existait donc une suite de points $P_n \to P_0$ telle que

$$\log \frac{1}{r_{P_0 P_n}} - g_n(P_0, P_n) < C \quad \ldots\ldots \quad (19)$$

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[1]) Voir note [1]), p. 48.

Rappelons maintenant que $g(P_0) = 0$ donc $\log\frac{k}{d} \leqq 0$. Il s'en suit $d \geqq k$, donc les domaines contiennent tous le cercle C_k de rayon $k > 0$ et de centre P_0. Par comparaison des valeurs-frontière et en appliquant le principe de maximum des fonctions harmoniques[1], on voit que chaque $g_n(P_0, P)$ (P intérieur à C_k) vaut au plus la valeur en P de la fonction g correspondant à C_k, c.a.d. vaut au plus la constante $\log\frac{1}{k}$. On est alors arrivé à une contradiction: le membre à gauche de (19) tendant vers infini si $n \to \infty$, ne peut rester $< C$. Il s'en suit même une borne inférieure pour $\varphi(C)$.

En effet, il résulte de (19):

$$\log\frac{1}{r} - \log\frac{1}{k} < C$$

donc

$$r > k\,e^{-C}$$

Il s'en suit

$$\varphi(C) \geqq k\,e^{-C} \quad \ldots\ldots\ldots \quad (20)$$

ou avec (18)

$$\varphi(C) \geqq \tfrac{1}{2}\,e^{-\frac{\pi}{2}-C}$$

Par une considération des domaines $G(P, P_0) > C$, on arrive à l'existence d'une fonction $0 < \psi(C) < \infty$. En effet, en supposant que $\psi(C) = \infty$, on peut trouver une suite de domaines $\{\Omega_n\}$ et une suite de points $\{P_n\}$, où P_n est dans Ω_n, telle que $P_n \to P_\infty$ et

$$\log\frac{1}{r_{P_0 P_n}} - g_n(P_0, P_n) > C.$$

Remarquons maintenant que $g_n(P_0, P_n) = g_n(P_n, P_0)$ de sorte qu'on peut prendre le pôle dans P_n. On voit alors que le membre à gauche de cette inégalité doit tendre vers zéro, pourvu que g_n ne tend pas vers une fonction identiquement égal à $-\infty$. Or ceci n'est pas le cas, les frontières Σ_n ayant tous des points intérieur au cercle de rayon 1 et de centre P_0, puisqu'on a supposé $g_n(P_0) = 0$. On est donc arrivé à une contradiction.

Le passage au cas où $g(P_0) \neq 0$ est maintenant simple. Appliquons pour cela la transformation (17). On a

$$\bar{g}(P_0) = g(P_0) + \log\frac{1}{p}.$$

En supposant $\bar{g}(P_0) = o$ et $g(P_0)$ égale à une valeur donnée, il faut prendre $p = e^{g(P_0)}$. La courbe $\bar{G} = C$ est transformée en $G = C$. La première courbe se trouve entre les cercles $(\varphi(C))$ et $(\psi(C))$, donc la courbe $G = C$ entre les cercles de rayon

$$\varphi(C)\,e^{-g(P_0)} \quad \text{et} \quad \psi(C)\,e^{-g(P_0)}.$$

Des tentatives pour déterminer les valeurs exactes de φ et de ψ, comme données dans l'introduction, par les méthodes précédentes n'avaient pas encore de résultat.

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1) Remarquons que $g(P, P_0) < \log\frac{1}{r_{PP_0}}$.

Biochemistry. — *Tissues of prismatic cells containing Biocolloids.* IV. *Morphological changes of the complex coacervate gelatine + gum arabic in consequence of a pH change of the medium flowing along the membrane.* By H. G. BUNGENBERG DE JONG and B. KOK. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of November 29, 1941.)

1. *Introduction.*

In this and in the next communication we shall discuss the effect of the pH, of some neutral salts and non-electrolytes on a complex coacervate formed in the prismatic cells of a celloidin membrane. Some morphological changes were observed which — in view of what we know of the effect of the variables mentioned on the water percentage of the complex coacervate — are unexpected. These changes are *vacuolization processes* (i.e. de-mixing of new equilibrium liquid from the coacervate) *in spite of the fact that the water percentage of the coacervate increases.*

2. *Methods.*

The methods employed are in principle like those described previously [1]. As in communications I and III a solution of 6 g. gum arabic + 5 g. gelatine + 200 g. water was enclosed in the celloidin membrane. The cuvette used is a modification of that of fig. 2 in the first communication. Instead of one tube there are two, which, by means of two thin flexible rubber tubes are connected with two glass reservoirs with taps. The complex coacervation is brought about by causing 0.01 N acetic acid to flow along the membrane.

When the coacervate has become parietal and practically free from vacuoles (occasionally except for a few large ones), we change on to the second reservoir, which contains a different solution of acetic acid or a solution of a neutral salt, respectively a non-electrolyte in 0.01 N acetic acid. The ensuing morphological effects "*inflow effects*" are observed for some time until the picture practically ceases to change, after which we return to the first reservoir (0.01 N acetic acid) and the morphological effects caused "*outflow effects*" are again observed for some time. After this the membrane was always removed and replaced by a fresh one. It is true that the same membrane may be used a few times in succession, but owing probably to the imperfect impermeability of the celloidin membrane the character of the inflow and the outflow effects gradually changes when the cycle is repeated several times. In order therefore to obtain comparable results a membrane is used only once for an inflow and outflow cycle.

In order to replace the original medium as quickly as possible by another one, the clearance was reduced to a minimum by cementing a glass cube in the centre of the cuvette (between membrane and glass cube an opening of ca. 1 mm is left for the flowing medium).

Moreover, in order to prevent complications in consequence of gelation of the complex coacervate, care should be taken that the temperature in the cuvette is above ca. 33° (preferably between 35° and 40°).

We desist from a detailed description of the experiment apparatus, only noting that *a.* the cuvette is sunk in a copper box through which hot water is led, *b.* that a similar heating box is mounted round the objective of the microscope, *c.* the medium liquid enters the cuvette at a temperature of ca. 40° and *d.* the medium liquid leaving the cuvette drips on to a thermometer, so that it can always be ascertained whether the temperature is still at least 35°.

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1) See Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, XLIII, 512, 732 (1940).

3. *Charge condition of the complex coacervate enclosed in the cells when 0.01* N *acetic acid is led past them.*

In the explanation of the in- and outflow effects on variation of the pH, which will be dealt with in 8., the charge condition of the complex coacervate in the original medium is of great significance. The colloid mixture enclosed in the membrane (6 g. gum arabic + 5 g. gelatine + 200 g. H_2O) has been chosen in such a way that the complex coacervate is practically uncharged when 0.01 N acetic acid flows past (pH 3.35). This is apparent from the electrophoretic direction *a* of the coacervate drops in the celloidin cells a short time after they are formed and before they coalesce to a parietal coacervate, *b* the little vacuoles still enclosed in the parietal coacervate (observable for a short time only, disturbances occurring later on in consequence of polarization of the celloidin walls). On coacervation with acetic acid solutions of different concentration this electrophoretic direction is indicative of a strongly negative charge with pH 3.85, 3.76, 3.65, 3.55, of a weakly negative charge with pH 3.45, a weakly positive charge with pH 3.26 and a strongly positive one with pH 3.05, 2.96, 2.8 and 2.65. With pH 3.35 the direction of the motion of the coacervate drops was uncertain, mostly, however, pointing to a weakly negative charge, like the behaviour of the vacuoles. These qualitative observations therefore indicate that the reverse of charge takes place between pH 3.35 and 3.26, and much nearer to the first value than to the second.

Below are given electrophoresis measurements made at 38° in a microscopic cuvette of mixtures consisting of 100 cc acetic acid of varied concentration + 1 cc of the stock solution (6 g. gum arabic + 5 g. gelatine + 200 g. H_2O), which lead to the same conclusion. Moreover the survey gives the results of another, similar series of measurements made in the constant presence of 10 m aeq. p. L., KCl (significant for the in- and outflow effects with KCl containing acetic acid, which will be discussed in the next communication).

Acetic acid conc. in N.	pH	U (arbitrary units)	
		Without salt	With KCl 10 m aeq.
0.030	3.11	+ 333	+ 171
0.025	3.15	+ 275	+ 138
0.020	3.20	+ 194	+ 92
0.015	3.26		
0.008	3.40	— 138	— 122
0.006	3.45	— 238	— 148
0.005	3.50	— 325	— 215
Reversal of charge (graphically interpolated)		pH 3.32	pH 3.28

So we see that the 0.01 N acetic acid used (pH 3.35) in the colloid proportion given must indeed cause a practically uncharged coacervate in the celloidin cells.

The question may be asked why in the above electrophoretic measurements the gelatine-gum arabic mixture was diluted 100 ×. The reply is this: If we should measure colloid systems of considerably greater concentrations, complications would arise which are absent in the celloidin membrane in the coacervated system and which are also practically avoided when the colloid mixture is greatly diluted. These complications are the result of the salt formed in the complex coacervation from the counter ions of the two colloids. In this case the salt formed is Ca acetate, which may interfere in two ways: *a.* by shifting the point of reversal of charge, *b.* by a pH change, the acetate together with the acetic acid forming a buffer system. On complex coacervation in the cells of the membrane the salt formed from the counter ions of the two colloids is washed away and so these complications do not arise. On complex coacervation in vitro, however, the salt formed remains in the system and these complications can only be neglected when the colloid system — and with it the salt concentration — is made very small.

4. *Expectations concerning the nature of in- and outflow effects on increase or decrease of the* pH, *based on the data of 3.*

We know from previous investigations that there is an intimate correlation between condition of charge, water percentage of the coacervate and colloid percentage of the equilibrium liquid, namely so, that with the uncharged complex coacervate the colloid percentage of the equilibrium liquid is minimal and — apart from certain complications — the water percentage of a complex coacervate is also minimal. Since we have seen in 3. that the coacervate formed with 0.01 N acetic acid (pH 3.35) is practically uncharged, we may expect that the increase as well as the decrease of the pH will increase both the colloid percentage of the equilibrium liquid and the water percentage of the coacervate.

Hence no morphological effects may be expected from inflow. The coacervate must remain free from vacuoles, similarly no new coacervate drops may form in the vacuole. On outflow — return to the original pH — on the other hand, the parietal coacervate must vacuolize and new coacervate drops must form in the central vacuole. We shall see in 5. that actually there are important deviations from these expectations.

5. *Inflow and outflow effects on variation of the* pH.

a. Owing to the variation of the acetic acid concentration.

When the original 0.01 N acetic acid is replaced by 1/30 resp. 1/300 N acetic acid, there are practically no changes[1] in the original picture. So there are no inflow effects of a morphological nature. On subsequent replacement by 0.01 N acetic acid outflow effects are likewise absent.

When the original 0.01 N acetic acid is replaced by 1/10 N acetic acid vacuolization occurs in the parietal coacervate. When after inflow of sufficient duration we return to 0.01 N the vacuolization in the coacervate persists, at most decreases slightly, new coacervate drops forming in the large central vacuole. After sufficient time these drops coalesce with the parietal coacervate, this becoming free from vacuoles.

When the original 0.01 N acetic acid is replaced by 1/10 N acetic acid vacuolization also occurs in the parietal coacervate. On change to 0.01 N acetic acid the same happens in principle as described above with 0.1 N acetic acid, the number of coacervate drops forming in the central vacuole only being smaller. Occasionally we observed that besides the vacuoles originally present, new small vacuoles were formed in the parietal coacervate.

Figure 1 is a schematic summary of the above observation.

b. Owing to variation of the HCl *concentration.*

The question may be asked whether the changes described under *a.* are specific effects of acetic acid or if they are the consequences of pH variation. If the latter is the case it must also be possible to obtain them with isohydric HCl solutions. If for the calculation of the pH of the acetic acid solutions employed we assume pK = 4.7 for acetic acid, it follows that of 0.1 N = pH 2.85; 0.033 N = pH 3.09; 0.01 N = pH 3.35; 0.0033 N = pH 3.59; 0.001 N = pH 3.85.

Isohydric HCl solutions were prepared and with them the experiments described in *a.* were repeated. The method of working is as follows. A membrane — each time new-prepared — is washed first with 0.01 N acetic acid (pH = 3.35), until the final condition is reached, then we change to an isohydric HCl solution (0.45 millimol HCl/L) and thus the acetic acid is washed out, no changes occurring in the morphological picture. Then we change to a HCl solution of a different pH and after sufficient duration we return to the HCl solution of pH 3.35. The following results were obtained:

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[1] With 1/300 N acetic acid some very small vacuoles only form in the large cells of the coacervate, which persist on washing with 1/100 N acetic acid.

pH 2.85	inflow:	strong vacuolization in the parietal coacervate.
	outflow:	the vacuolization decreases slightly, new coacervation drops form in the central vacuole.
pH 3.11	inflow:	no changes.
	outflow:	no changes.
pH 3.61	inflow:	as in the case with acetic acid, practically no changes.
	outflow:	no changes.
pH 3.85	inflow:	slow vacuolization of the parietal coacervate.
	outflow:	vacuolization persists, few small, new coacervate drops form in the central vacuole.

The results are very much like those obtained with acetic acid, so that we are warranted in ascribing the morphological changes to the variation of the pH of the medium led past the membrane.

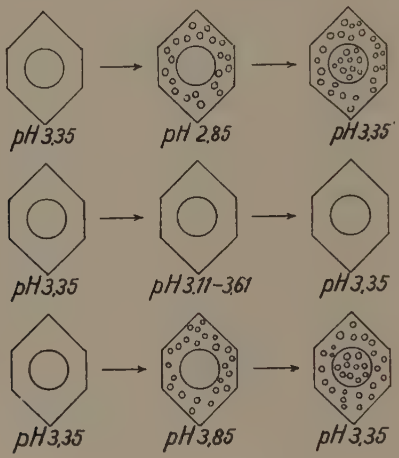

Fig. 1. In- and outflow effects with a medium liquid of lower or higher pH.

6. *Preliminary discussion. The experiment is not in accordance with the expectations.*

When we compare the in- and outflow effects observed with the expectations pronounced in 4., we see that the two do not tally. The absence of any morphological effect on slight pH variations is not so significant, as this may be in consequence of the fact that the displacements of components — which are now also slighter — may take place sufficiently rapidly by diffusion, without causing any new morphological phenomena. So we have to observe the pH changes (of ½ pH unit) which do occasion in- and outflow effects.

The only effect expected which does take place at the expected moment is the formation of new coacervate drops in the central vacuole on outflow. It is true that after outflow the parietal coacervation shows vacuolization, so apparently in accordance with our expectation, but vacuolization had already occurred during inflow, which had not been foreseen.

We must therefore try to find a reason why on increase or decrease of the pH vacuolization of the coacervate occurs, although the water percentage increases at the same time.

7. *New data concerning the influence of the* pH *on complex coacervates.*

The non-accordance stated in 6. made us presume that the knowledge we possess concerning the changes in composition of complex coacervates as f (pH) — which we employed in stating our expectations in 4. —, is incomplete.

The complex coacervation of gelatine and gum arabic has formerly been studied extensively [1]. Then a.o. we analyzed the coacervates and equilibrium liquids which are formed on modification of the mixing proportions of isohydric gelatine and gum arabic sols. The results, however, cannot be directly applied in an explanation of the in- and outflow effects, because here we work with a constant mixing proportion (the mixture of sols enclosed in the cells of the celloidin membrane) and the pH is varied.

Indirectly, however, the results mentioned may be used, owing to the fact that we examined mixing series with five different pH values, which are comparable in every respect. If we base the analysis for one constant mixing proportion on these mixing series, we obtain data which relate to a variation of the pH at constant mixing proportion.

The only mixing proportion occurring in all five mixing series is that of 50 % gum arabic, or rather, in mixing proportions with a very slight variance (between 49.9 and 50.6 % A), but very near 50 % gum arabic.

When these are selected, therefore, we obtain the most complete data, which moreover we have corrected graphically for exactly 50 % gum arabic.

These corrected data are given in the survey and set out graphically in fig. 2.

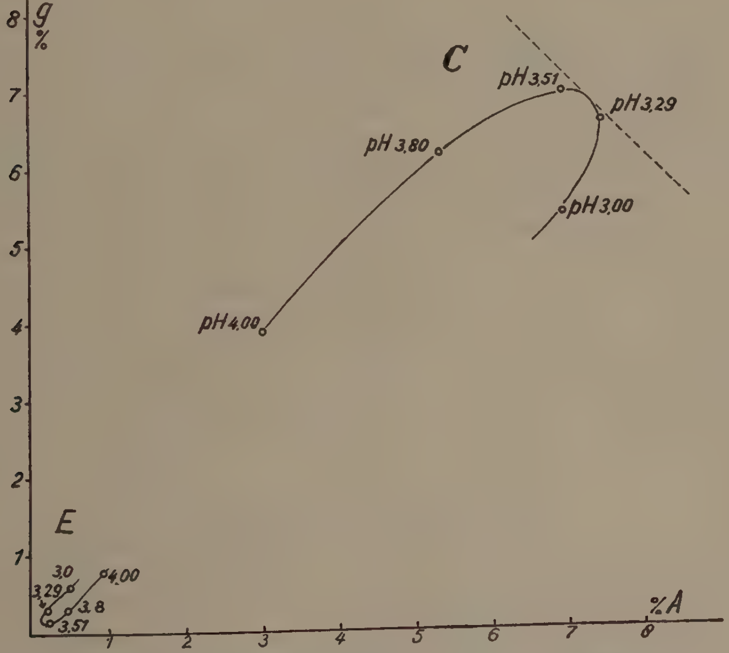

Fig. 2. Change in the composition of coacervate (*C*) and equilibrium liquid (*E*), with pH, at constant colloid proportion in the total system.

The following five points become apparent, of which one and two are in accordance with the premises employed in the deduction of the expectations in 4.:

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[1] H. G. BUNGENBERG DE JONG and W. A. DEKKER, Koll. Beih. **43**, 213 (1936). The analysis results mentioned further in this text are mentioned in that article on p.p. 222 and 223.

1. At a certain pH — here ca 3.4 — the dryweight of the coacervate (i.e. A + G, that is the gelatine + gum arabic percentage), attains a maximum, i.e. the water percentage of the coacervate reaches a minimum (in Fig. 1 at the spot where the dotted line drawn at an angle of 45° touches the upper curve).

2. At practically the same pH the colloid percentage of the equilibrium liquid is also minimal (in Fig. 1 obtainable analogously by drawing a tangent to the lower curve at an angle of 45°).

3. The relative proportion of the two colloids in the coacervate is changed at a variation of the pH and the coacervate becomes relatively richer in gum arabic when the pH decreases.

4. The relative proportion of the two colloids in the equilibrium liquid is changed in a pH section round the pH mentioned in 1. and 2. at first in a reversed sense.

5. The proportion of the two colloids in the coacervate at the pH mentioned in 1. and 2. is equal to that in the equilibrium liquid.

pH	Coacervate		Equilibr. liquid		% (A + G)		% A / % G	
	% gelatine	% gum ar.	% gelatine	% gum ar.	Coacervate	Equilibrium liquid	Coacervate	Equilibrium liquid
3.00	5.44	6.92	0.59	0.495	12.36	1.085	1.27	0.84
3.29	6.68	7.39	0.277	0.183	14.07	0.46	1.11	0.66
3.51	7.06	6.92	0.175	0.195	13.98	0.37	0.98	1.11
3.80	6.25	5.27	0.324	0.446	11.52	0.77	0.84	1.38
4.00	3.9	3.0	0.80	0.90	6.9	1.70	0.77	1.12

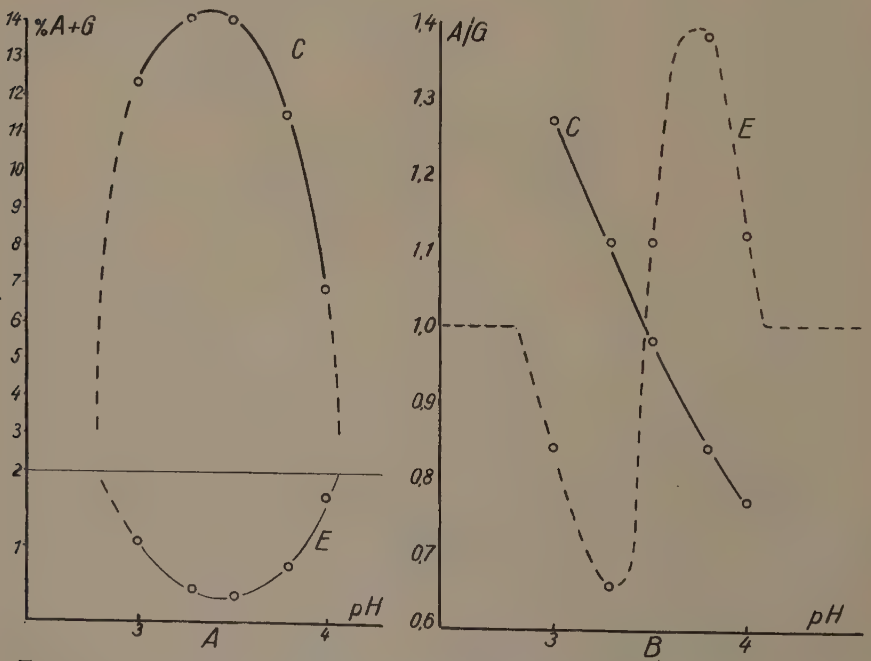

Fig. 3. A. Change of the colloid percentage of the coacervate (C), resp. of the equilibrium liquid (E) with the pH.

B. Change of the colloid proportion in the coacervate (C), resp. in the equilibrium liquid (E) with the pH.

These five points become more apparent in the separate graphs of fig. 3 (and in the schemes of figure 4), in which curves C apply to the coacervate and E to the equilibrium liquid.

8. *Explanation of the inflow and outflow effects.*

When in a given mixing proportion of the two colloids in the total system the pH has been selected so that the coacervation is optimal we are at the points given in fig. 4, namely at the maximum of curve C and the minimum of curve E in fig. 4a and at the point of intersection of curves C and E in fig. 4b.

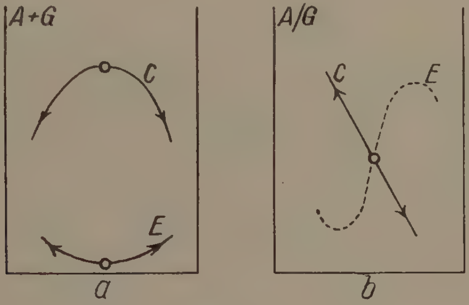

Fig. 4.

According to what has been said in 3. the complex coacervate enclosed in the celloidin cells is in this condition at pH 3.35 (optimal coacervation takes place practically at the point of reverse of charge). For the explanation of the in- and outflow effects we may start immediately from these schemes. Arrows indicate the direction in which during the inflow with a liquid of different pH the working point shifts along curves C and E of fig. 4A and along curve C of fig. 4B. As for the interpretation of the in- and outflow effects the consideration of the change of the A/G proportion in the equilibrium liquid does not open new aspects, curve E in fig. 4B is dotted and no arrows are inserted. On outflow the shifted working point moves in opposite direction along the curves to its original place.

We will now see what morphological changes may be the consequence of a shifting of the working point along curves C and E of fig. 3A and along curve C of fig. 4B and it will become clear that a summation of the changes considered separately before is in accordance with the in- and outflow effects observed.

So the three shiftings show:

I. Increase of the water percentage in the coacervate.
II. Increase of the colloid percentage in the equilibrium liquid.
III. Modification of the colloid percentage in the coacervate.

The results of I. and II. need not be discussed at length, they have already been treated in 4. and together they show: absence of inflow effects and on outflow they show vacuolization of the parietal coacervate and formation of new coacervate drops in the central vacuole.

The surprising result that already on inflow vacuolization of the parietal coacervate occurs, although the coacervate becomes richer in water may now be understood by taking into account the change of the colloid proportion in the coacervate (III). On increase of the pH the gum arabic percentage for instance decreases relatively as regards the gelatine.

This change under simultaneous change of the composition of the equilibrium liquid must take place throughout the coacervate.

With the slight diffusion velocity of the colloids this can only occur without any morphological changes with a thin lamella of coacervate lying directly against the central vacuole. The rest of the coacervate, however, must then vacuolize i.e. the change in the colloid proportion takes place here under the formation in the parietal coacervate of new vacuoles, containing equilibrium liquid.

When, therefore, we observe the colloid proportion in the coacervate it becomes clear that the inflow effect consists in vacuolization, in spite of the fact that the coacervate becomes richer in water at the same time.

We wonder what the effect will be on outflow. On return to the original pH the proportion of the two colloids will return to the original one. Here too we must consider whether the colloid displacement between coacervate and equilibrium liquid may be effected sufficiently rapidly by diffusion only.

As for this, however, we are in a more favourable position than with inflow, because now the parietal coacervate layer is full of vacuoles (containing equilibrium liquid). When for a moment we leave out of consideration the fact that on return to the original pH the coacervate becomes poorer in water, so when we exclusively observe the change of the colloid proportions, two possibilities are seen to exist:

a. the colloid displacement by diffusion is sufficiently rapid, in which case there will be no new vacuolization and so the picture caused on inflow will be retained,

b. the colloid displacement by diffusion is not rapid enough, in which case a new generation of vacuoles must be formed in addition to those formed on inflow.

A summation of the three effects discussed I, II and III may lead to the expectation that the total effect on inflow will be vacuolization in the parietal coacervate and on outflow that the coacervate remains vacuolized (new vacuoles may even form), new coacervate drops forming in the central vacuole.

These expectations are in accordance with the in- and outflow effects observed.

SUMMARY.

1. We studied the morphological changes in consequence of pH variations of an uncharged complex coacervate enclosed in the cells of a celloidin membrane.

2. Vacuolization of the parietal coacervate takes place both on sufficient pH increase and decrease.

3. On return to the original pH the coacervate at first remains vacuolized, while new coacervate drops are formed in the central vacuole.

4. Considering that the uncharged complex coacervate becomes richer in water on increase as well as on decrease of the pH, 2. is unexpected.

5. On pH change the colloid proportion in the coacervate is also modified, as is seen from new data concerning the effect of the pH on the composition of a coacervate and equilibrium liquid at constant proportion of the colloids in the total system.

6. The morphological changes mentioned in 2 and 3 may be understood from the summation of three effects resulting from pH decrease or increase:

a. the increase of the water percentage of the coacervate,

b. the increase of the colloid percentage in the equilibrium liquid,

c. the modification of the colloid proportion in the coacervate.

Biochemistry. — *Effect of neutral salts on the composition of complex coacervate (gelatine + gum arabic) and equilibrium liquid at constant* pH *and constant mixing proportion of the two colloids in the total system.* By H. G. BUNGENBERG DE JONG and E. G. HOSKAM. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of November 29, 1941.)

1. *First method of investigation.*

Although the question put in the title can only be answered directly by analysis, the experiments rescribed in the following pages enable us to answer the question indirectly. We followed two methods, which led to the same conclusion.

The first method is the simplest experimentally, as we avoid the preparation of isohydric gelatine and gum arabic sols, necessary in the second method.

It makes use of the fact that when in a number of mixing proportions of gelatine and gum arabic which are constant within each series, the coacervate volume is determined as a function of the added quantity of HCl, the coacervate volume curve is generally asymmetrical, becoming symmetrical with a certain mixing proportion.

Starting from purified gelatine [1]) and gum arabic [2]) we made 2½ % (air dry) mixed stock solutions, such that the proportion of gelatine and gum arabic changes mutually. This mixing proportion we express in what follows in % *A* (= gum arabic) of the colloid mixture.

In sedimentation tubes provided at their lower ends with narrower cylindrical tubes divided into 0.1 cc, is pipetted 10 cc stock solution and then a cc HCl 0.1 N + (2.5 — a) cc H_2O. After mixing the tubes are placed in the thermostat at 40° and after one night the coacervate volume is noted.

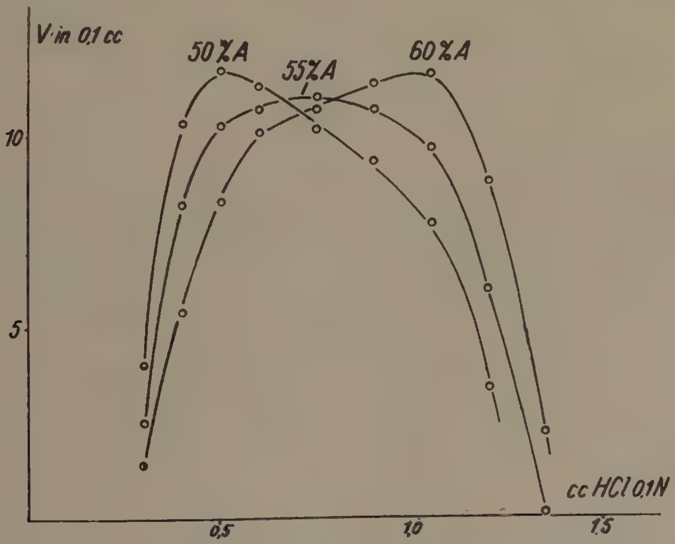

Fig. 1. Shape of the coacervate volume curves in some mixing proportions of the two colloids in the total system.

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[1]) F00 extra of the "Lijm- en Gelatinefabriek 'Delft'" at Delft, purified by a method described previously (Kolloid Beihefte **43**, 256, 1936), a modification of LOEB's method.

[2]) Gomme Sénégal petite boule blanche I, of Allan et Robert, Paris.

In Fig. 1 are shown the coacervate volume curves for 50, 55 and 60 % *A*. We see that the ones for 50 % *A* and 60 % *A* are very asymmetrical, but that for 55 % *A* has its maximum (0.74 cc HCl 0.1 N) practically in the centre of the HCl concentration section, in which coacervation takes place.

These maxima (determinable by the construction of bisecting lines, see Fig. 2) have shifted considerably to the left in the case of 45 and 50 % *A* (0.46, resp. 0.53 cc HCl 0.1 N), and to the right in the case of 60 and 65 % *A* (0.88 resp. 1.0 cc HCl 0.1 N).

For the investigation of the effect of neutral salts we select a colloid mixture (55 % *A*) whose curve is practically symmetrical with HCl and now we examine whether in the presence of salt the coacervate volume curve becomes asymmetrical.

We shall not here discuss all the experimental material, restricting ourselves to a single instance, see Fig. 2, namely a comparison of a blank series with series in which 6 m aeq. p. L. $Co(NH_3)_6Cl_3$ resp. $K_3CH(SO_3)_3$ are present, but in which otherwise the final concentration of the colloids in the mixtures are practically the same as in Fig. 1 (we always use 5 cc 55 % *A* containing stock sol with a total colloid concentration of 5 %, to 12.5 cc final volume, so that there is 7.5 cc left for HCl and salt solution).

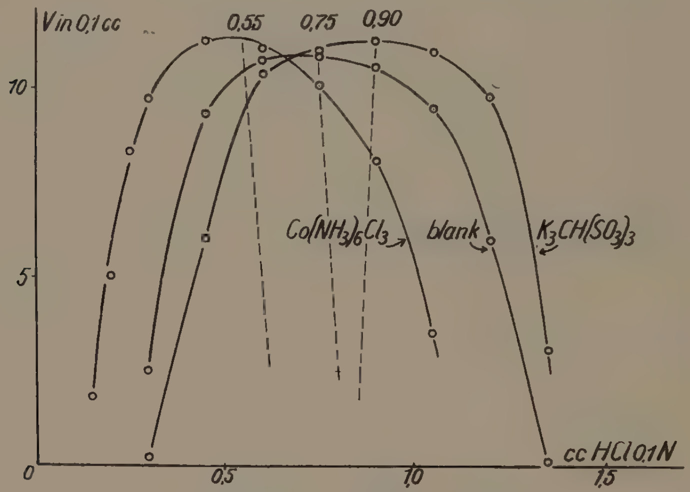

Fig. 2. Effect of some salts on the shape of the coacervate volume curve with constant mixing proportion of the two colloids in the total system.

With the aid of bisecting lines in Fig. 2 we find that the maximum is shifted by $Co(NH_3)_6Cl_3$ to smaller added HCl quantities (0.55 cc), by $K_3CH(SO_3)_3$ to greater ones (0.90 cc) than is necessary in the blank series (0.75 cc).

In this way we measured for 7 salts, each time at 3 concentrations, coacervate volumes as function of the added quantities of HCl, and the position of the maximum was determined graphically. Figure 3 shows the results obtained, from which we see that as regards the shifting of the maximum the salts arrange themselves in the series:

(Increasing pH) 3 — 1 ... 2 — 1... 1 — 1 ...1 — 2 ...1 — 3 (decreasing pH).

KCl has here no influence (type 1 — 1) and 2 —1 and 3 — 1 shift the pH of optimal coacervation to higher, 1 — 2 and 1 — 3 to lower values.

We would also point out that the order in which the salts are arranged here is the same as occurs in the so-called *"continuous valence rule"*:

(relative positivation) 3 — 1 ... 2 — 1 ...1 — 1 ...1 — 2...1 — 3 (relative negativation)

This rule applies to the effects of salts on the electrophoretic velocity of complex coacervate drops (at constant pH and mixing proportions of the colloids in the total system) [1].

This rule should not be confused with the so-called *"double valence rule"*:

$$3 - 1 > 2 - 1 > 1 - 1$$
$$1 - 3 > 1 - 2 > 1 - 1$$

which applies to the neutralizing effect of neutral salts on complex coacervation [1]. The experimental material necessary for the above experiments shows for a constant mixing proportion (55 % *A*) and a constant added quantity of HCl (0.75 cc 0.1 N) the following neutralization concentrations (coacervation volume = 0):

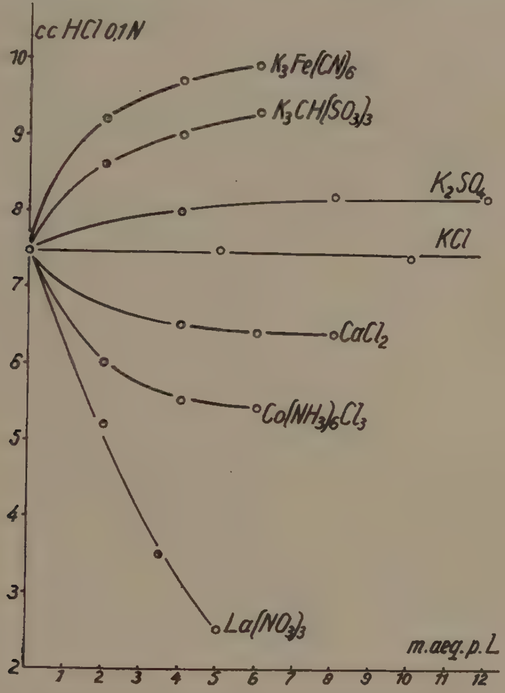

Fig. 3. Shifting of the maximum of the coacervate volume curve (in cc 0.1 N HCl) owing to salts.

	$K_3Fe(CN)_6$	$K_3CH(SO_3)_3$	K_2SO_4	KCl	$CaCl_2$	$CO(NH_3)_6Cl_3$	$La(NO_3)_3$
Neutralization with m. aeq. p. L.	9	19	28	33	18	9	6

Also in this neutralization the two salts of type 3 — 1 and the two salts of 1 — 3 do not act with equal strength, so that here besides the valence of the ions other factors should be considered [2].

2. *Second method of investigation.*

In the second method of investigation the experiments are arranged in such a way that we modify the mixing proportion at constant pH, this enabling us to determine the shifting of the mixing proportion of optimal coacervation.

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[1] H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Kolloid. Beihefte **43**, 143 (1935).

[2] Probably they may be ascribed to coëffects: thus with $K_3Fe(CN)_6$ the coacervate layers and equilibrium liquids become rapidly bluish-green and $La(NO_3)_3$ causes a decrease of pH (at 5 m. aeq. p. L. the pH of our systems decreases from ca. 3.7 to 3.4).

We then made separate stock solutions of the two colloids (5 g air dry + 100 cc dist. water) and determined first for them pH titration curves at 40°, preparing mixtures of the following composition: 20 cc stock solution + *a* cc 0.1 N HCl + (30 — *a*) cc H_2O. From these curves we could see how much HCl was to be added in order to prepare isohydric gelatine ($a = 3.8$) and gum arabic ($a = 2.1$) sols of pH 3.70. We made two such sols, only taking 10 cc H_2O less, the final volume not being 50 cc but 40 cc.

In sedimentation tubes we placed 5 cc water resp. 30 m aeq. salt solution, then **b** cc gum arabic solution and (20 — *b*) cc gelatine sol.

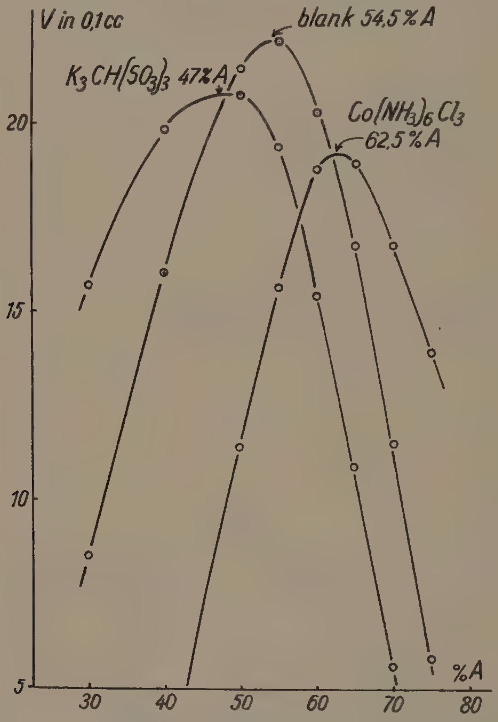

Fig. 4. Shifting of the optimal mixing proportion of the isohydric sols owing to 6 m. aeq. p. L. $Co(NH_3)_6Cl_3$ resp. $K_3CH(SO_3)_3$.

The final concentration of the colloids is now the same as in the determination of the pH titration curves, so that in this way we realize isohydric mixing series. On three successive days we compared in this way blank II — $Co(NH_3)_6Cl_3$ — $K_3CH(SO_3)_3$; blank I — KCl — K_2SO_4 and $CaCl_2$ — $La(No_3)_3$ — $K_3Fe(CN)_6$. The two blank series were slightly different in the absolute values of the coacervate volume, but (construction of a bisecting line) they give practically the same values for the mixing proportion of optimal coacervation (54 resp. 54.5 % *A*).

Table I contains the results of these experimental series, Fig. 4 showing the curves of a blank series and of the series with 6 m. aeq. $Co(NH_3)_6Cl_3$ and $K_3CH(SO_3)_3$.

Leaving again out of consideration the mutual difference between the two salts of type 3 — 1 and those between 1 — 3, we find for the intensity and direction of the shifting of the optimal mixing proportion the series:

(% *A* increases) 3 — 1... 2 — 1 ...1 — 1...1 — 2...1 — 3 (% *A* decreases)

i.e. the same order at which we had arrived by the first method of investigation.

Again KCl (1 — 1) has no influence here and the shifting owing to 3 — 1 and 2 — 1 is the opposite of that owing to 1—2 and 1—3

3. *Agreement of the results obtained by the two methods of investigation.*

By the first method of investigation in 1. we found that at constant mixing proportion (being that of optimal coacervation without salt) neutral salts shift the pH according to the *continuous valence rule*, polyvalent cations causing shifting to higher and polyvalent anions to a lower pH:

(increasing pH) 3—1...2—1...1—1...1—2...1—3 (decreasing pH).

From this it may be deduced directly how with constant pH salts affect the mixing proportion of optimal coacervation. With this in mind we will again consider Fig. 2. With the mixing proportion selected of 55 % *A* without salt the maximum of the coacervate volume curve is found to be at 0.75 cc HCl 0.1 N. Now the coacervates at pH values higher, resp. lower than those of the maximum are positively, resp. negatively charged, while the uncharged coacervate is found at or very near the pH of the maximum (corresponding to 0.75 cc HCl 0.1 N).

TABLE I.

Coacervate volumes (in 0.1 cc) in the absence and in the presence of 6 m. aeq. p. L. salt as function of the mixing proportion of the two isohydric sols (pH 3.70).

Mix. prop. isohydr. sols in % A	$K_3Fe(CN)_6$	$K_3CH(SO_3)_3$	K_2SO_4	KCl	Blank I	Blank II	$CaCl_2$	$Co(NH_3)_6Cl_3$	$La(NO_3)_3$
10	5.3								
20	11.6								
30	16.5	15.7	12.0	8.5	7.7	8.5	3.6	0	
35	18.2								
40	18.6	19.8	18.4	16.7	15.9	16.0	12.8	2.2	
45	17.7								
50	16.4	20.7	21.5	21.1	20.9	21.4	19.6	11.4	
55	13.5	19.3	21.4	21.8	21.4	22.1	20.5	15.6	0.7
60		15.4	18.8	20.5	20.8	20.2	20.6	18.7	4.1
65		10.9	15.2	17.6	16.8	16.7	18.9	18.8	7.6
70		5.6	9.7	12.2	12.0	11.5	14.4	16.7	9.5
75		1.6	4.3	6.5	6.0	5.8	9.7	13.9	9.5
80									7.0
85									4.4
90									1.4
Mix. prop. optimal coacerv.	39.5 % A	47 % A	51.5% A	54% A	54 % A	54.5 % A	56.5% A	62.5 % A	72 % A

Graphically, as discussed above, we determined the position of the maxima, given in the lowest horizontal row of the table.

With the latter pH we are, however, when $Co(NH_3)_6Cl_3$ is present, already on the right descending branch of the curve, i.e. there where the coacervate charge is positive. In order to bring the coacervate at the point of reversal of charge at this pH, we should have to add isohydrically a certain quantity of *A* (negative gum arabic solution) to the 55 % colloid mixture. From this it follows that in the presence of $Co(NH_3)_6Cl_3$ and at constant pH the mixing proportion of optimal coacervation has shifted to colloid mixtures richer in *A*.

Analogously we deduce that in the presence of $K_3CH(SO_3)_3$ and with constant pH the mixing proportion of optimal coacervation has shifted to colloid mixtures of lower *A* percentage.

From this it follows in conformity with the result in 2., that at constant pH salts must affect the mixing proportion of optimal coacervation as follows:

(optimal mix. prop. shifts to systems of higher A percentage)	3 — 1 ...2 — 1... 1—1...1—2...1 — 3	(optimal mix. prop. shifts to systems of lower A percentage)

in which 1 — 1 (KCl) does not affect the mixing proportion of optimal coacervation applying to the blank series.

4. *Conclusions for the change in composition of a complex coacervate owing to salts at constant* pH *and constant mixing proportion of the two colloids in the total system.*

From previous investigations 1) we know that — at least when no salt is added — the proportion of gelatine and gum arabic in the complex coacervate varies (with constant pH) with the mixing proportion of the two colloids in the total system. With the mixing proportion of optimal coacervation, the A/G proportion in the coacervate is equal to that of the equilibrium liquid and therefore to that in the total system.

With mixing proportions richer in A than those of optimal coacervation the coacervate also takes up A, but in such a way that the shifting of A/G in the coacervate is less than in the total system. Likewise the coacervate can still take up G from mixing proportions richer in G than those of the optimal mixing proportion, but again the shifting of A/G in the coacervate is less than in the total system.

So there is a certain tendency in the complex coacervate to maintain with constant pH the composition belonging to the mixing proportion of optimal coacervation.

When we remember these properties of the complex coacervates and presume that they persist in the presence of salts, we can deduce how the salts will change the A/G proportion in the coacervate at constant pH and constant mixing proportion of the two colloids in the total system.

Let us suppose that with constant mixing proportion in the total system we select that one which without salt is the optimal mixing proportion. In the presence of $K_3CH(SO_3)_3$ however, the mixing proportion given is no longer the optimal one, but it contains too much A (for the optimal mixing proportion is poorer in A than it was without salt). With the mixing proportion given the coacervate will still take up A, but a relatively limited quantity on account of the persistency discussed above. The consequence will be that in the presence of $K_3CH(SO_3)_3$ the A/G proportion in the coacervate will be smaller than that of the coacervate formed in the absence of salts. In the same way the conclusion is reasoned out that with the same pH and mixing proportion of the sols the A/G proportion in the coacervate formed in the presence of $Co(NH_3)_6Cl_3$ will be greater than in the coacervate formed in the absence of salt. Thus we arrive at the conclusion that the *continuous valence rule* must also be applicable to the modification of the colloid proportion in the coacervate when salts are added at constant pH and mixing proportion of the two colloids, namely:

(A/G incresae in the coacervate)	3 — 1...2 — 1...1 — 1...1 — 2...1 — 3	(A/G decrease in the coacervate).

As we are concerned with constant mixing proportion of A and G in the total system, A/G in the equilibrium liquid will necessarily shift in a reversed sense.

5. *Verification of the conclusion for the case of* $CaCl_2$.

During a previous investigation 1) of complex coacervation we also investigated the effect of $CaCl_2$ on the composition of coacervate and equilibrium liquid (the complex coacervation of gelatine and arabic acid). The investigation really aimed at verifying the increase of the water percentage of a complex coacervate owing to neutral salts, which assumption had been based on different grounds. While the investigation confirmed

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1) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, loc. cit.

this supposition an additional effect of $CaCl_2$ was noted, namely the shifting of optimal mixing proportion to mixtures of a higher *A* percentage (so in accordance with 2).

From these analysis results we can also see if — in accordance with what has been discussed in 4. — at constant pH and mixing proportion in the total system, the complex coacervate gains relatively in *A* by the addition of $CaCl_2$, so whether the equilibrium liquid necessarily becomes poorer in *A*. The following Table (II) shows the results for the mixing proportion of 45 % *A*.

TABLE II.

$CaCl_2$ m. aeq. p. L.	Coacervate			Equilibrium liquid			A/G	
	% A + G	% G	% A	% A + G	% G	% A	coacervate	equilibrium liquid
0	15.90	8.96	6.94	0.39	0.23	0.16	0.77	0.70
5	12.81	7.06	5.72	0.84	0.51	0.30	0.81	0.59
7.5	11.94	6.40	5.50	1.04	0.63	0.37	0.86	0.59 ¹)

The table first shows the general effect of a neutral salt: the water percentage of the coacervate increases (% $A + G$ decreases) and the equilibrium liquid becomes richer in colloids (% $A + G$ increases). Moreover it is seen from the last two columns that A/G does indeed increase in the coacervate and that A/G in the equilibrium liquid decreases. The change of $A + G$ as well as of A/G are also clearly visible in Fig. 5, in which the

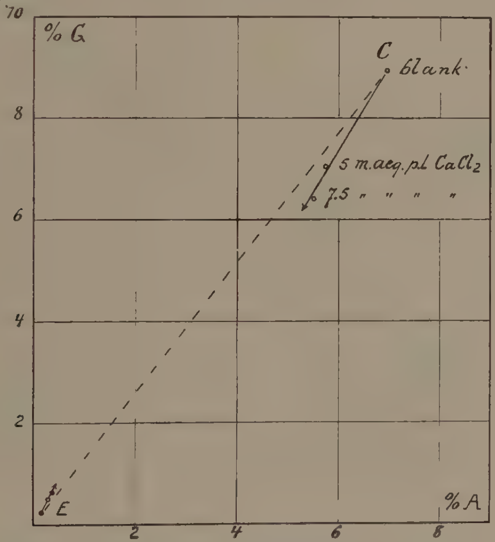

Fig. 5. Effect of $CaCl_2$ on the composition of coacervate (*C*), resp. equilibrium liquid (*E*).

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¹) The material gathered in the table was taken from three separate experimental series. Owing to the fact that in each of these series the isohydric gelatine and arabic acid sols were of a slightly different concentration, irregularities such as the equality of A/G in the equilibrium liquid for 5 and 7.5 m aeq. $CaCl_2$ should not be considered real, although these figures are not equal in the coacervate. As a matter of fact the analysis figures of the equilibrium liquid are always much less accurate than those of the coacervate. In all the other mixing proportions (31.6; 35; 40; 50; 51.6 %) we shall always see that the $CaCl_2$ increases the proportion of A/G in the coacervate, but from the analysis figures of the equilibrium liquid we see deviations in some mixing proportions from the expected decrease of A/G, which we cannot, however, consider real.

analysis figures (% *A* and % *G*) of Table II are set out. In this connection we should notice the courses of the points representing the composition of coacervate (*C*) and equilibrium liquid (*E*).

If these points should move towards each other only along the dotted connecting line, this would only prove that the water percentage of the coacervate increases ($A + G$ of the coacervate decreases), likewise that the colloid percentage of the equilibrium liquid increases, without any change being brought about in the proportion of A/G in the coacervate.

The figure shows that the course of the coacervate point deviates downwards from the connecting line, that the course of the point of the equilibrium liquid deviates upwards from the connecting line. This means that A/G increases in the coacervate but that it decreases in the equilibrium liquid.

The case discussed of $CaCl_2$ is the only one in which analysis of the *A* and *G* percentages were made, so that we do not dispose of further material to verify the conclusions drawn in the previous section. On the ground of those conclusions we may expect that for salts 1 — 1 the coacervate and the equilibrium liquid points will move towards each other, approximately along the connecting line, that for a salt 3 — 1 the deviations will be in the same direction as for $CaCl_2$ (2 — 1) but greater, that for a salt 1 — 2 and even more for a salt 1 — 3 the deviation from the connecting line will be the reverse of those for 2 — 1. Compare scheme Fig. 8.

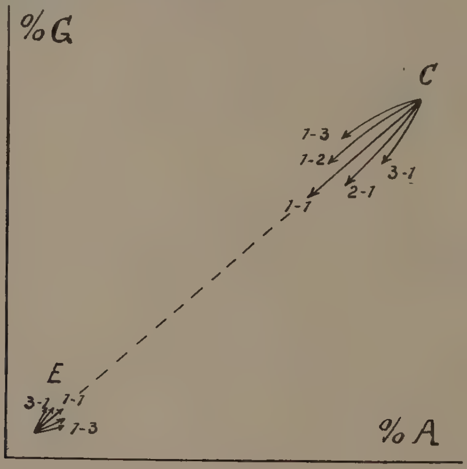

Fig. 6. Scheme of the effect of neutral salts on the composition of coacervate and equilibrium liquid.

Summary.

1. At constant pH and constant mixing proportion of the colloids (gelatine, gum arabic) in the total system the addition of salts causes a change, not only of the water percentage in a complex coacervate, but also in the colloid proportion in the coacervate.

2. The *continuous valence rule* is applicable to the change of the colloid proportion, namely: 3 — 1...2 — 1...1 — 1...1 — 2...1 — 3, in which 1 — 1 does not modify the proportion, 2 — 1 and even more 3 — 1 increase the gum arabic percentage of the coacervate, whereas 1 — 2 and even more 1 — 3 increase the gelatine percentage of the coacervate.

3. The proportion of the two colloids in the equilibrium liquid is modified in a reversed sense from that in the coacervate.

Biochemistry. — *Tissues of prismatic cells containing Biocolloids.* V. *Morphological changes of the complex coacervate gelatine-gum arabic owing to the addition of salts resp. non-electrolytes to the liquid flowing past the membrane.* By H. G. BUNGENBERG DE JONG and B. KOK. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of November 29, 1941.)

Introduction and methods.

In the previous communication we discussed the effect of a pH change [1]). In this communication follow our results as regards the effect of neutral salts and of some non-electrolytes. For colloid preparations employed and apparatus we refer to that communication.

As regards the method, we first brought about coacervation in a newly prepared membrane by conducting past it 0.01 N acetic acid, then we changed to a solution of a salt, resp. non-electrolyte in 0.01 N acetic acid, observing the *inflow effects.* Finally in order to study the *outflow effects* 0.01 N acetic acid is again conducted past the membrane, after which the membrane is removed and replaced by a freshly prepared one.

I. Inflow and outflow effects with neutral salts.

a. Effect of neutral salts on the complex coacervate.

We know that salts have a neutralizing effect on complex coacervation and the more strongly (i.e. with smaller concentrations) in proportion as with unchanging valence (e.g. monovalent) of the cation the valence of the anion increases, likewise when with unchanging valence of the anion the valence of the cation increases. We again checked this rule in our colloid preparations measuring the volume of the coacervate, which demixes on otherwise equal conditions on variation of the salt concentration.

In sedimentation tubes whose cylindrical lower ends were narrowed and divided into 0.1 c.c., we placed 1 cc HCl 0.1 N + a cc salt solution + (6.5 + a) dist. water.

After placing in the thermostat at 40° we added to each tube 5 cc sol mixture (6 g. gum arabic + 5 g. gelatine + 190 cc H_2O), the contents of the tube were mixed and then left in the thermostat. The next morning the coacervate volume was read (estimated in 0.01 cc). The results are given in the following table:

TABLE I. Coacervate volume in 0.1 cc in the presents of salts Blank = 13.1.

Concentr. m. aeq. p. L.	K_3FeCy_6	K_2SO_4	KCl	$CaCl_2$	$La(NO_3)_3$
2	12.0	—	—	—	12.5
4	10.5	13.0	13.1	12.8	9.5
6	7.7	—	—	—	2.7
8	5.3	12.2	12.9	10.4	0
10	3.3	—	—	—	
12	0	10.8	11.9	4.5	
16		8.1	10.8	0	
20		3.9	7.6		
24		0	1.8		
28			0		
Neutralization conc. in m. aeq. p. L.	ca 12	ca 22	ca 25	ca 14	ca 7

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[1]) H. G. BUNGENBERG DE JONG and B. KOK, Proc. Ned. Akad. v. Wetensch., Amsterdam, 45, 51 (1942).

Graphically the neutralization concentrations mentioned in the lowest horizontal row of the table are found, which proves the validity of the valence rule mentioned:

$$\begin{array}{l} 3-1 > 2-1 > 1-1 \\ 1-3 > 1-2 > 1-1 \end{array} \quad \text{"Double valence rule"}$$

This rule may be foreseen from the screening effect of the cation on the arabinate-colloid anion and of the anion on the gelatine-colloid cation.

As the screening effect increases with the valence of the ions, and the mutual electric attraction of the two colloidions possesses the character of a product, it follows that for neutralization of the complex coacervation the double valence rule must apply. It is further to be expected that *the water percentage of the coacervate must increase in salts concentrations preceding the neutralization.*

This conclusion had formerly been confirmed by analyses in the case of the effect of $CaCl_2$ on complex coacervates of gelatine and arabic acid sols, the results of which we give here (left) for an arbitrary mixing proportion (50 %). From an investigation made lately as to the effect of the temperature we can also take material in confirmation of this conclusion for KCl. There are given the results for an arbitrarily selected temperature (40°) and constant mixing proportion and pH (right).

TABLE II.

Effect of $CaCl_2$			Effect of KCl		
Salt conc. in m. aeq. p L.	% A+G coacervate	% A+G equil. liquid	Salt conc. in m. aeq. p. L.	% A+G coacervate	% A+G equil. liquid
0	15.24	0.35	0	13.20	0.50
5	13.05	0.65	5	12.02	0.69
7.5	12.00	0.85	10	10.90	0.96
			20	8.84	1.63

In the two tables we see that the dryweight of the coacervate decreases (water percentage increases) and the dryweight of the equilibrium liquid increases on increase of the salt concentration. The mutual mixability of the two liquids (coacervate and equilibrium liquid) increases therefore as we approach the neutralization concentration.

b. *Expectations as to the nature of inflow and outflow effects based on a).*

As neutral salts increase the water percentage of the coacervate, while the colloid percentage of the equilibrium liquid also increases, we cannot expect any morphological changes on inflow. On outflow of the salt the waterpercentage of the coacervate decreases again, like the colloid percentage of the equilibrium liquid. Here we can indeed expect morphological changes, namely vacuolization of the parietal coacervate and formation of new coacervate drops in the large central vacuole (which contains the equilibrium liquid).

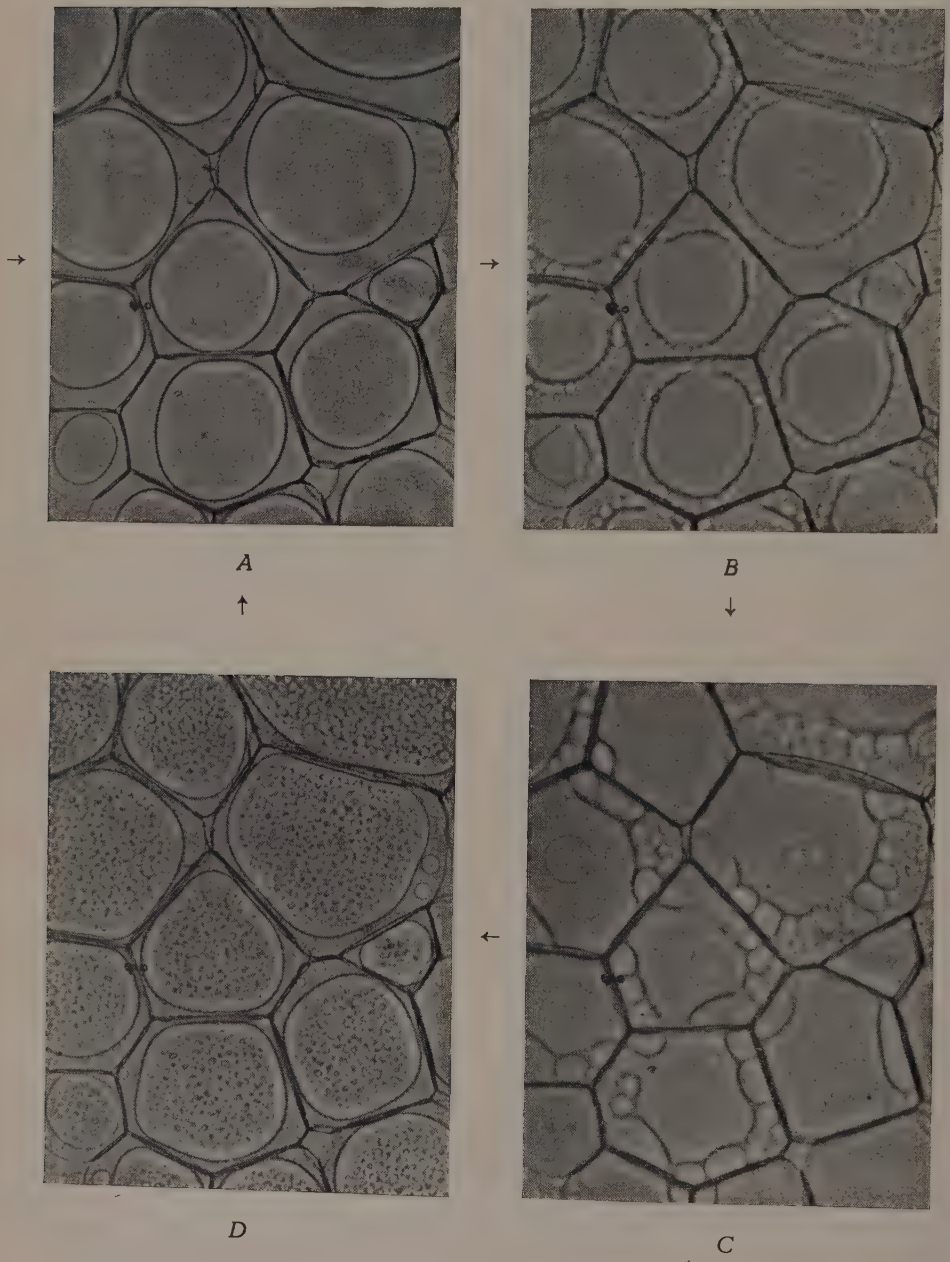
c. *Inflow effects. Distinction of four concentration sections.*

It is found experimentally that for each salt four concentration sections may be distinguished, depending on the nature of the morphological processes on inflow. Based as these sections are on the appreciation of microscopic pictures, their limits cannot be indicated. Of course, they pass into each other without any clear demarcation. We distinguish sections:

- *α.* in which no morphological effects occur;
- *β.* in which equally divided vacuolization of the coacervate takes place;
- *γ.* in which the vacuolization is clearly localized or at least begins to appear in certain places of the coacervate;
- *δ.* in which the coacervate entirely dissolves. The following survey gives the salt

H. G. BUNGENBERG DE JONG and B. KOK: Tissues of prismatic cells containing Biocolloids. V.

PLATE 1.



In- and outflow cycle with $K_3Fe(CN)_6$.
A: Initial state.
B and *C*: Inflow effect.
D: Outflow effect.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLV, 1942.

concentrations investigated in milli-aequivalent pL arranged in four columns, corresponding to the concentration sections mentioned.

TABLE III.

Salt	α No vacuolization	β Homogeneous vacuolization	γ Localised vacuolization	δ Neutralization	Neutralization concentration cp. *a*
$K_3Fe(CN)_6$	0.3	0.75—1.5—3	5—10	15	12
K_2SO_4	1.5	3—5—10—20	(25)	25	22
KCl	5	10—15	20	25—30—40	25
$CaCl_2$	1—2	5	10	15	14
$Co(NH_3)_6Cl_3$	1	3	5	10	—
$La(NO_3)_3$	1	2.5—5	—	—	7

From the survey it is clear that the neutralization concentrations have the same order of magnitude for the complex coacervate enclosed in the celloidin cells as found in *a*.

It is further noticeable that the boundaries between sections α/β resp. β/γ lie at higher concentrations as the salt has greater difficulty in neutralizing the coacervate. These boundaries rise in the order $K_3Fe(CN)_6$—K_2SO_4—KCl, similarly in the order $La(NO_3)_3$ or $Co(NH_3)_6Cl_3$—$CaCl_2$—KCl.

This suggests a connection between the morphological changes on inflow and the neutralizing effect of neutral salts (Double valence rule). But this leads to a contradiction, as neutral salts at concentrations preceding neutralization increase the water percentage of the coacervate, so that no vacuolization is to be expected.

Concentration sections α and δ are not the most interesting to us. With the relatively small salt concentrations in α the attendant internal changes in the composition of the coacervate can apparently be sufficiently brought about by diffusion so that vacuolization is not enforced.

In section δ, morphological processes belonging in γ precede the neutralization in the cases in which in γ the vacuolization processes are very intense and rapid (e.g. $K_3Fe(CN)_6$). When the changes in γ are slower and less intense they are absent in δ. This is the case with KCl at 30 and 40 m. aeq. p. L. with $CaCl_2$ at 15 m. aeq. and with $Co(NH_3)_6Cl_3$ at 10 m. aeq. p. L. So in these cases there is no vacuolization in the parietal coacervate. The central vacuole is generally seen to become rapidly smaller and the bounding face: central vacuole coacervate, which at first was clearly visible is soon obscured. All this is accounted for by the now reversed proportion of the tempo and the intensity of the neutralization and vacuolization processes.

d. Effect of the nature of the salt on the character of the in- and outflow effects.

An instance of marked deviation from the expectations mentioned in *b.* is furnished by the inflow and outflow effects with $K_3Fe(CN)_6$, (salt type 1—3), which will be discussed in connection with four microphotographs of Plate I.

A. Shows a part of the celloidin membrane after the complex coacervate formed with 0.01 N acetic acid has become entirely parietal and free from vacuoles.

B. Shows that after a short period of inflow with 10 m.aeq. $K_3Fe(CN)_6$ containing 0.01 N acetic acid, there is *vacuolization of the parietal coacervate* and that this process sets in along the bounding face coacervate/central vacuole. This vacuolization process now becomes very intensive, extending over the entire parietal coacervate. The vacuoles become larger, flatten each other so that a foam structure is formed. Many foam lamellae burst so that a relatively small number of large foam vacuoles are left.

C. Shows such a foam structure after some time of inflow, after the foam-forming processes have practically ceased.

D. Shows the condition after a short period of outflow with 0.01 N acetic acid. The foam lamellae burst and *no vacuoles are formed in the parietal coacervate* 1), while a great number of new coacervate drops are formed in the central vacuole. These gradually coalesce with each other and with the parietal coacervate, so that if one waits long enough one sees again the picture of microphotograph *A.*

The picture given in *D* is not different in any other way from the stage also found in the original coacervation with acetic acid 0.01 N and which gradually led to the condition in *A.* So in cycle *A—B—C—D—A* the absence of coacervation in the central vacuole during inflow and the process of coacervation in the central vacuole on outflow are in conformity with the expectation of *b.* The vacuolization of the parietal coacervate on inflow and practically its absence on outflow, however, are not in accordance with what was expected.

A picture entirely differing in many details is given by the in- and outflow cycles with $La(NO_3)_3$ or with $Co(NH_3)_6Cl$ (Salt type 3—1) while the other salts form a gradual transition between these two extremes (1—3 and 3—1) arranging themselves in the series:

$$\begin{array}{ccccccccc} K_3Fe(CN)_6 & — & K_2SO_4 & — & KCl & — & CaCl_2 & — & Co(NH_3)_6Cl_3 \text{ resp. } La(NO)_3 \\ (1—3) & \ldots & (1—2) & \ldots & (1—1) & \ldots & (2—1) & \ldots & (3—1) \end{array}$$

Compare fig. 1, which gives a scheme of the actual points of difference between $K_3Fe(CN)_6$, KCl and $Co(NH_3)_6Cl_3$.

The following changes occur from left to right in this salt series:

1. The velocity and the intensity of the vacuolization on *inflow* decrease considerably.

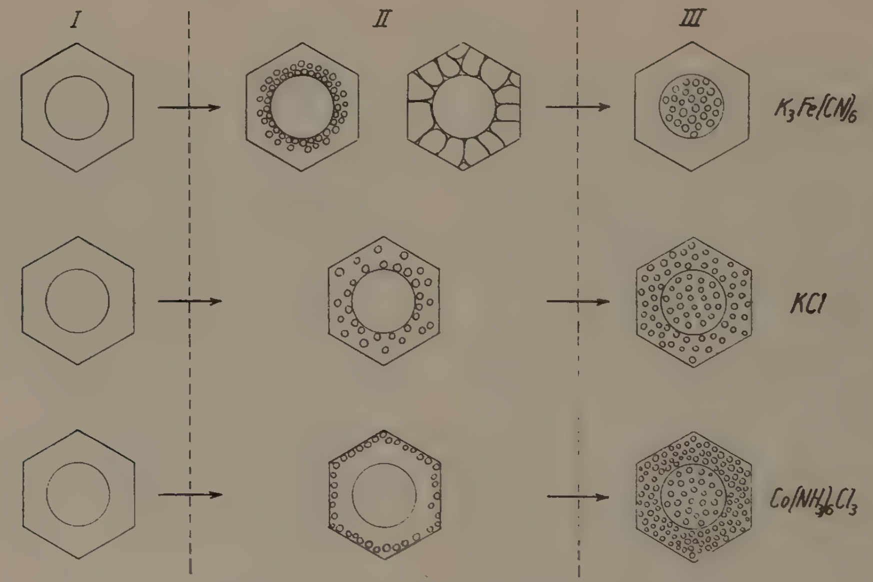


Fig. 1. In- and outflow effects with $K_3Fe(CN)_6$, KCl and $Co(NH_3)_6Cl_3$.
I. initial state.
II. inflow effects.
III. outflow effects.

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1) On outflow there are some indications of very weak vacuolization. A great number of very small points is formed (probably vacuoles) which, however, soon disappear.

With 2—1 and especially with 3—1 for instance, vacuolization is very slight and slow.

2. The localization of the vacuoles first appearing on inflow in concentration section γ changes gradually. With $K_3Fe(CN)_6$ and K_2SO_4 they occur round the central vacuole, with $CaCl_2$ and $Co(NH_3)_6Cl_3$ on the other hand, round the walls of the cell. In KCl, where this localization process is rather vague, it is perhaps like $K_3Fe(CN)_6$ and K_2SO_4.

3. Secondary changes of the vacuoles formed on inflow soon recede to the background from left to right. Foam formation is very marked with $K_3Fe(CN)_6$, with K_2SO_4 it is possible at most to speak of a passing tendency to foam formation, any other indication with the other salts being absent.

4. The velocity and especially the intensity of vacuolization on *outflow* increase considerably from left to right in the series. With $K_3Fe(CN)_6$ vacuolization is practically absent. With K_2SO_4 it is already fairly noticeable, increasing considerably in the order of KCl—$CaCl_2$—$Co(NH_3)_6Cl_3$. On outflow vacuolization we have not been able to state with any certainty any details concerning localization resp. secondary changes of the vacuoles (analogous to points 2 and 3 above).

e. *Details concerning the formation and disappearance of vacuoles.*

When the vacuolization is rapid, there are generally at first a great many small vacuoles, whose number usually decreases fairly rapidly owing to mutual coalescence or to coalescence with the central vacuole (the latter process is much retarted with foam formation).

When vacuoles have formed on inflow, and one does not wait till they have all disappeared, these which remain often undergo changes of diminishing volume and (or) number on outflow. This is most evident in small vacuoles.

When inflow is begun with a parietal coacervate which has not yet become entirely free from vacuoles, it is seen that with certain salts (KCl, $CaCl_2$, $Co(NH_3)_6Cl_3$) these vacuoles become smaller or disappear in concentration section α.

All this might be taken as an argument in favour of the "neutralizing effect" of neutral salts, viz. an increase of the water percentage of the coacervate. But such an accelerated disappearance of vacuoles is also seen on outflow after inflow with certain other salts ($K_3Fe(CN)_6$, K_2SO_4, possibly also KCl). In accordance with the expectations in *b.* we may expect vacuolization on outflow, because the coacervate becomes poorer in water. There is no reason therefore for the disappearance of vacuoles which have formed on inflow.

It is even the rule, that when either on inflow, or on outflow there is vacuolization, vacuoles which had formed owing the previous process, nevertheless disappear, while simultaneously or shortly afterwards new vacuolization occurs. We will moreover mention the fact that a new generation of vacuoles arises ever deeper in the membrane, i.e. on the side of the coacervate which is turned towards the medium liquid flowing past the membrane.

Finally we mention that vacuolization processes, especially when they are not intense, are much more noticeable in the larger cells of the celloidin membrane than in small ones. In large cells the rate of the exchange of matter between coacervate and central vacuoles is retarded by the greater thickness of the parietal coacervate layer. A slight change in composition which can be brought about in the small cells with sufficient rapidity by diffusion while the coacervate remains homogeneous, will be too slow in larger cells, so that this change is now brought about under the formation of vacuoles.

f. *Discussion.*

While describing the in- and outflow effects we have repeatedly pointed out, that the expectations expressed in *b.* based on the effect of salts on the water percentage of a complex coacervate, are not at all in accordance with the effects found experimentally.

There we are rather in the same position as we were in the previous communication in connection with the effect of the pH. There too we found vacuolizations on inflow, although the water percentage of the coacervate also increases. A solution of the difficulty was given by further observation of the simultaneous change of the colloid proportion in the coacervate.

Analogously the question may here be asked if the fact that on inflow with neutral salts vacuolization practically always occurs, is not to be ascribed to a coeffect of the salt, causing a change in the colloid proportion in the coacervate.

Such a coeffect of $CaCl_2$ on complex coacervates we had observed in a previous investigation of the complex coacervation of gelatine (positive) + arabic acid (negative) [1].

We did not know, however, if this coeffect is also active in the case of gelatine + gum arabic, and if it is also brought about by other neutral salts. As we must know this, however, in order to arrive at an interpretation of the in- and outflow effects described, we lately made some investigations, the result of which was that generally neutral salts cause a change in the colloid proportion of the coacervate with constant pH and constant mixing proportion of the colloids in the total system (these are the conditions under which the in- and outflow effects were studied) [2].

As regards intensity and direction this change depends on the nature of the salts, which arrange themselves in the order:

A/G in the coacervate increases 3—1 ... 2—1 ... 1—1 ... 1—2 ... 1—3 A/G in the coacervate decreases

in which 1—1 has practically no effect on *A/G* (= proportion of gum arabic and gelatine in the coacervate) [3].

This is exactly the same order of the salts which we found above in the description of the in- and outflow effects (see *d*).

This takes away in principle the apparent inconsistency that vacuolizations occur on inflow in spite of the fact that the coacervate can only become richer in water in consequence of added salts. For owing to the change of the colloid proportion a certain quantity of *A.* resp. *G.* must moreover leave the coacervate and as the diffusion velocity of the colloids is only slight, this occasions expulsion in the form of vacuoles (in which there separates the new equilibrium liquid belonging to the new colloid composition). Some details in e. can now also be accounted for, e.g. the disappearance on outflow of some vacuoles belonging to the inflow generation and the simultaneous or subsequent formation of a new vacuolization. The small vacuoles of the inflow generation embedded in the coacervate disappear then on outflow, because their location is very favourable for reversibility (the colloid containing equilibrium liquid originally expelled is taken up again in the coacervate, the latter using it to recover its original colloid proportion).

The new generation of vacuoles which form on outflow is then to be ascribed to the diminishing of the waterpercentage of the coacervate.

Although, therefore we understand in principle the formation of vacuoles on inflow with salts, there is not yet full accordance. It is true that in the in- and outflow effects we also find the series:

3—1 ... 2—1 ... 1—1 ... 1—2 ... 1—3,

but we did not find that the inflow vacuolization in this series decreases from 3—1 to 2—1, that it is absent or very weak with 1—1 to increase via 1—2 and 1—3.

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[1] H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Kolloid Beihefte **43**, 213 (1936).

[2] H. G. BUNGENBERG DE JONG and E. G. HOSKAM, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 59 (1942).

[3] Compare in communication IV of this series the very slight effect of KCl on the pH with reversal of charge of the complex coacervate.

This was to be expected from the result of the investigation cited, in which we saw that *A/G* in the coacervate increases with 3—1 and 2—1, is constant with 1—1 and decreases with 1—2 and 1—3.

We rather get the impression that for the coacervate in the celloidin membrane it is not salt 1—1 which causes the least change in the *A/G* proportion in the coacervate, but that it is salts 2—1 and 3—1 which have that effect.

For the in- and outflow effects caused by these salts come nearest the expectations given in *b* (where we did not take the *A/G* change into account); the inflow vacuolization is very weak here and on outflow there is strong vacuolization of the parietal coacervate and formation of new coacervate drops in the central vacuole.

It is not impossible that for the coacervate in the celloidin membrane the shifting of the point of neutralization in the salt series has been moved from 1—1 to 2—1 or to a place between 2—1 and 3—1. For this coacervate is not quite comparable with the one we examined in sedimentation tubes ("vitro"). In the latter we always retain in the total system the Ca salt which is formed from the counterions of the two colloids (Ca ions of gum arabic and Cl, resp. acetate ions of gelatine). But in the membrane this is removed by the medium liquid which flows continually past it. Since, as regards the main problem: the formation of vacuoles on inflow with salts we have yet found a satisfactory solution, there is no point in trying to find an explanation of the many other details observed (e.g. the location of the vacuoles).

We only note that the evident foam formation with $K_3Fe(CN)_6$ is reminiscent of analogous foam formations previously studied, for which we found that they depend on negativation [1]). In this connection we would point out that relative positivation and relative negativation is also brought about with salts and that to this the long-known, so called "continuous valence rule" is applicable, in which the order of the salts is the same as in the series discussed:

relative positivation 3—1 ... 2—1 ... 1—1 ... 1—2 ... 1—3 relative negativation

With the complex coacervate gelatine-gum arabic we found in communication IV (see Table on p. 68) of this series that ("in vitro") KCl is again very near the neutralization point in this series. $K_3Fe(CN)_6$ (1—3) in this series of salts is therefore the strongest negativing salt and probably connected with this in the fact that here the vacuoles formed on inflow undergo the secondary changes mentioned (foam formation).

II. In- and outflow effects with some non-electrolytes.

a. *Effects of glucose on the complex coacervate and expectations based on it for the nature of in- and outflow effects.*

With the aid of the coacervate volume method we made ourselves acquainted with the effect of glucose (see above Ia). But here we did not work with constant, but with varying pH. Three experimental series were set in, in which we started from a 55 %. A containing sol mixture as employed in *b*. The composition of the mixture in these 3 series was:

A. 5 cc 55 % stocksol + 5 cc dist. H_2O + *a* cc HCl 0.1 N + (2.5—*a*) cc dist. H_2O.
B. 5 cc 55 % stocksol + 5 cc 25 % glucose + *a* cc HCl 0.1 N + (2.5—*a*) cc dist. H_2O.
C. 5 cc 55 % stocksol + 5 cc 50 % glucose + *a* cc HCl 0.1 N + (2.5—*a*) cc dist. H_2O.

When the results are set out graphically (Table IV) it is seen that 10 % resp. 20 % glucose:

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[1]) H. G. BUNGENBERG DE JONG and O. BANK, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 274 (1939).
H. G. BUNGENBERG DE JONG, O. BANK und E. G. HOSKAM, Protoplasma **34**, 30 (1940).

a. slightly increases the maximum of the coacervate volume curve;

b. clearly, but not greatly diminishes the section of the added quantities of HCl (so the pH section) in which coacervation takes place.

Increase of the maximum of the coacervate volume curve and change of the coacervation section are phenomena known form a previous investigation to be characteristic of the neutralizing effect of an agent ¹).

From these preliminary experiments we conclude that *10 % and 20 % glucose increase the waterpercentage of the coacervate.*

So the expectations for the nature of the in- and outflow effects wtih glucose are the same in principle as we expressed for salts in I *b*: No morphological effects on inflow and if the increase of the waterpercentage in consequence of the glucose has been sufficient, vacuolization of the parietal coacervate and formation of coacervate drops in the great central vacuole on outflow.

TABLE IV.

Effect of glucose on the coacervate volume (in 0.1 cc).

cc HCl 0.1 N	Blank	10 % glucose	20 % glucose
0.3	0.7	0	0
0.4	7.3	7.7	6.3
0.5	11.6	11.9	11.9
0.6	13.0	13.4	13.6
0.9	13.2	13.6	13.7
1.0	12.8	12.9	12.8
1.2	9.5	7.8	6.1
1.4	0	0	0

b. In- and outflow effects with glucose.

Glucose is practically inactive with concentrations which in the case of electrolytes cause considerable effects (10 milli mol). We must here choose the concentrations much higher to observe notable effects e.g. 10 % glucose (= 5/9 molary): so there is vacuolization of the parietal coacervate in which the vacuoles are preferably formed near the bounding face coacervate/central vacuole. Outflow causes vacuolization which is more intensive and in which the many vacuoles formed rapidly coalesce to a smaller number of larger vacuoles. On outflow there are no changes in the central vacuole.

The microphotographs (see plate II) show a similar cycle:

A. original coacervate formed with 0.01 N acetic acid;

B. inflow effect with 10 % glucose containing 0.01 N acetic acid;

C and *D.* successive stages on inflow with 0.01 N acetic acid. In these microphotographs there is one cell (right upper corner, with circular vacuole), which clearly demonstrates that on inflow the diameter of the central vacuole becomes a little smaller and that on outflow it first becomes smaller again (*C*) and then larger (*D*).

Certain matters of detail, observed with salts also occur with glucose e.g. that on outflow vacuolization small vacuoles belonging to the inflow generation first or simultaneously become smaller or disappear. These effects are clear, however, when the glucose concentration is taken somewhat smaller (5 or 2½ %).

c. In- and outflow effects with other non-electrolytes.

We note here that the same in- and outflow effects occur with aequimolecular solutions (5/9 mol p.L.) of saccharose, glycerine and ethylalcohol. The intensity of the effects greatly decreases in the order glucose — glycerine — alcohol.

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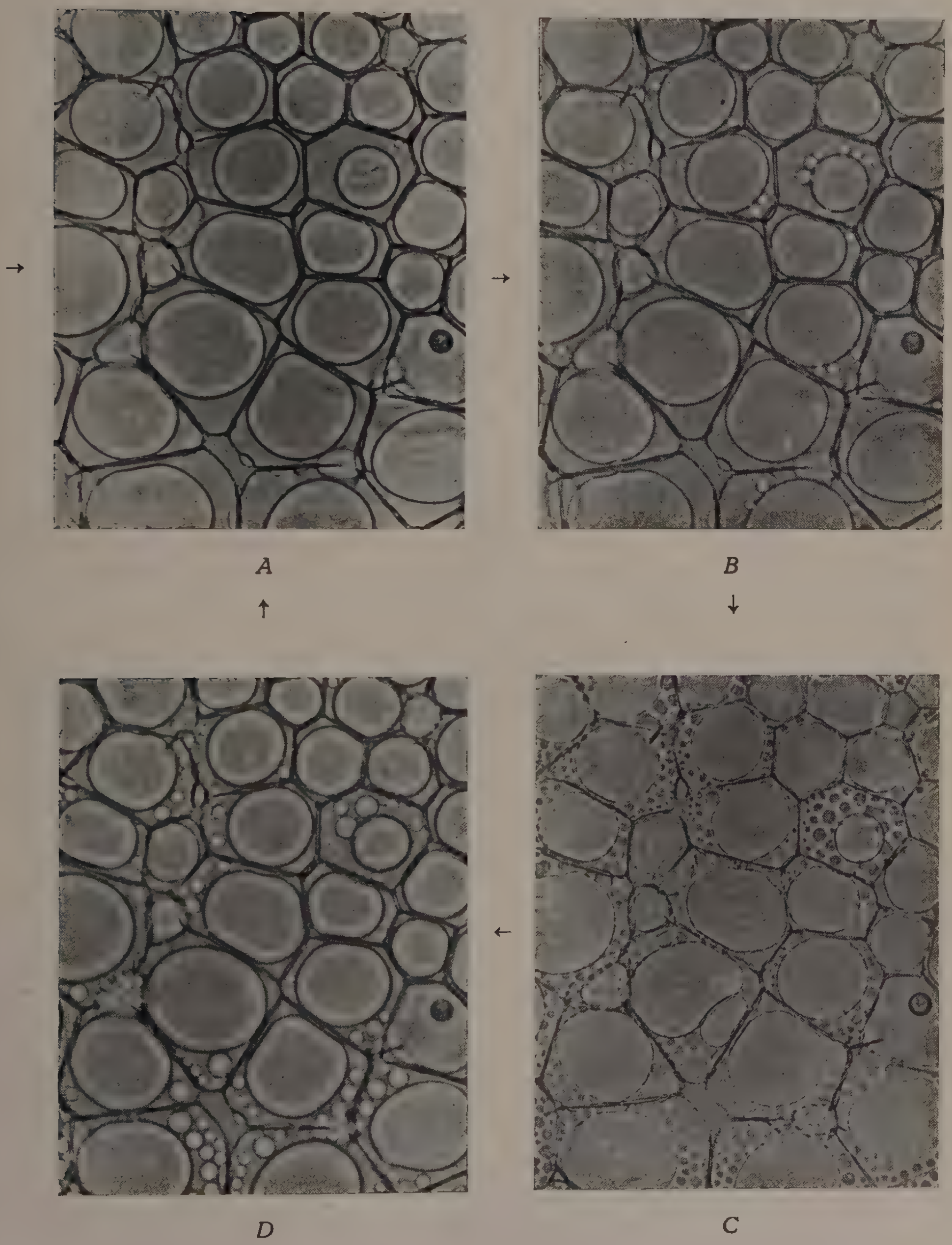
¹) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, loc. cit. compare p. 249, Table XI.

PLATE 2


A

B

C

D

In- and outflow cycle with glucose.

A: Initial state.
B: Inflow effect.
C and *D*: Outflow effect.

d. Discussion.

As in part I we are confronted with the same difficulty in the case of glucose; there is vacuolization on outflow, although this was not expected on the ground of *a*. So again we are inclined to think of a coëffect of the glucose in consequence of which the *A*/*G* proportion in the coacervate is changed. We are however not in the possession of data, nor in the case of saccharose and glycerine.

With ethylalcohol the case is different; it increases the colloid percentage of the coacervate, also causing as evident shifting in the *A*/*G* proportion in the coacervate. So there are here two reasons for vacuolization on inflow. Nevertheless this vacuolization is weak, for which fact we cannot account.

Summary.

1. The morphological changes are studied of an uncharged complex coacervate enclosed in the cells of a celloidin membrane, in consequence of the addition of salts resp. non-electrolytes to the medium (0.01 N acetic acid) flowing past the membrane and accompanying the removal of these substances by outflow with 0.01 N acetic acid.
2. Taking into consideration that the waterpercentage of the uncharged complex coacervate is increased by salts, resp. by glucose, the occurrence of vacuolization on inflow is unexpected.
3. In the case of salts there is formation of coacervate drops in the central vacuole on outflow. This phenomenous is in accordance with what is expected.
4. As regards the vacuolization intensity on inflow, the location of this vacuolization and the intensity of the vacuolization on outflow, the salts arrange themselves according to the so-called *continuous valence rule*:

   1—3 ... 1—2 ... 1—1 ... 2—1 ... 3—1.

5. New data concerning the effect of salts on the compositions of coacervate and equilibrium liquid with constant proportion of the colloids in the total system and with constant pH, show that salts can also modify the colloid proportion in the coacervate, namely according to the continuous valence rule. This also explains (2) in principle and the occurrence of the continuous valence rule in (4).
6. Many morphological matters of detail remain unexplained, however the occurrence of foam structures with $K_3Fe(CN)_6$ is probably attributable to strong negativation of the coacervate.

Leiden, Laboratory for Medical Chemistry.

Biochemistry. — *Tissues of prismatic cells containing Biocolloids.* VI. *Location of coëxisting coacervates and equilibrium liquid in the cells. Morphological model of the plant cell.* By H. G. BUNGENBERG DE JONG. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of November 29, 1941.)

1. *Introduction.*

After enclosing a mixture of gelatine and gum arabic sols in the celloidin membrane, coacervation occurs in the cells of the celloidin membrane, when an acid medium liquid is conducted past it. In the final condition we find that the complex coacervate has become parietal and that the equilibrium liquid has collected in one large "vacuole" surrounded by the coacervate [1]). This relative location is reminiscent of the analogous location of cell wall-cytoplasm-central vacuole in the mature plant cell. In the previous communications we have nearly exclusively occupied ourselves with the properties of this object of study (behaviour in the electric field, effect of a variation in the composition of the medium flowing past).

For biologists however, a somewhat more complicated object of study, consisting of two coëxisting coacervates and equilibrium liquid enclosed in the cells of the celloidin membrane might be even more interesting. Previous investigations [2]) have shown that from a suitable selected mixture of gelatine + gum arabic + Na-yeast nucleinate coëxisting coacervates are formed after acidification to a certain pH. These coacervates are distinct from each other because the one contains mostly nucleic acid besides gelatine (and a little gum arabic) the other containing mainly gum arabic besides gelatine (and a little nucleic acid). It appears that after the coacervation the drops of the two coacervates do not float in the equilibrium liquid loose from each other, but that those of the complex coacervate of high nucleinate percentage are taken up in those of the coacervate of high arabinate percentage. This location is analogous to that of nucleus, cytoplasm and surrounding medium in monocellular animal objects.

We first asked ourselves the question what will be the position with regard to each other of the two coacervates and the equilibrium liquid when we occasion complex coacervation of the mixture of the three sols (gelatine, gum arabic, nucleinate) in the cells of the celloidin membrane. Next analogously as in the complex coacervate gelatine + gum arabic we should study the properties of this object. In what follows we mainly communicate our experiences concerning the first question and further we shall discuss why for the time being it is not possible to carry out a deeper systematic investigation.

2. *Difficulties in connection with the enclosure of yeast nucleinate and some other colloids in the celloidin membrane.*

Whereas the method followed thus far for enclosing colloids in the cells of a celloidin membrane yields good results with gum arabic, with gelatine and with a mixture of the two, this is not the case with Na-yeast nucleinate, K-chondroitine sulphate and Clupeine. It is true that it is possible to enclose these colloids, but the celloidin membrane which encloses the cells on all sides is more or les permeable to them, so that they diffuse to the medium liquid flowing past the membrane with the result that the cells are soon free from colloids.

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[1]) H. G. BUNGENBERG DE JONG and B. KOK, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 67 (1942). Compare Plate I A.

[2]) H. G. BUNGENBERG DE JONG und A. DE HAAN, Biochem. Ztschr. **263**, 33 (1933).

As is to be expected these colloids also enter the cells when they are dissolved in the flowing medium liquid.

This is evident from the coacervation phenomena in the cells when only gelatine has been enclosed in the membrane and e.g. 0.01 N acetic acid containing very little nucleinate (e.g. 0.1—0.01 %) is conducted past the membrane. When after that only 0.01 N acetic acid is conducted past the membrane the coacervation gradually disappears. From these experiments it appears that the permeability of the celloidin membrane is not due to the presence of Na nucleinate during the formation of the membranes.

Nevertheless the nucleinate has a certain effect on the appearance of the membranes; the cells are not so large and visible defects are more numerous. Disturbances of this nature may cause the disintegration of the colloidin membrane when they become great. We know form experience that they may also proceed form fats and fatlike substances. The appearance of the membranes in which nucleinate has been enclosed improves indeed, when before enclosing it, the Na nucleinate solution is shaken in a separating funnel with CCl_4 which removes possible traces of fat. This preliminary process does not remove the principle difficulty; the permeability of the celloidin membrane to nucleinate.

We have in vain tried to remove the difficulty by the addition of various organic substances to the emulsification liquid (aether + amylalcohol + celloidin). Owing to this failure it is not as yet possible systematically to investigate nucleinate containing colloid mixtures whose mixing proportion is known, in the cells of the celloidin membrane. It is possible to obtain coacervation phenomena with them, but the enclosed system eventually loses nucleinate, more or less rapidly, so that the mixing proportion is continually changing in the direction of systems of lower nucleinate percentage.

3. *Complex coacervation of a mixture of gelatine, gum arabic and yeast nucleinate sols in the cells of the celloidin membrane.*

For the enclosing we used a mixed sol consisting of 3 g gelatine [1]), 1 g gum arabic [2]), 1 g Na-yeast nucleinate [3]), 250 cc dist. water.

This stock solution was first shaken with CCl_4. For enclosing we used a 2½ % solution of celloidin in amylalcohol + aether (1 : 1). For the further technique of the preparation of membranes [4]) and method of investigation we refer to previous communications [5]).

The best results were obtained by at a comparatively low temperature (the water leaving the cuvette is ca. 30°, so 2° to 3° higher in the cuvette) conducting past the membrane a buffer consisting of 10 m. aeq. p. L. Na acetate + 100 m. mol. p. L. acetic acid. Complex coacervation then occurs, of which a number of morphological pictures is given in fig. 1.

In fig. 1A the vacuolized complex coacervate of high nucleinate percentage forms a cohering mass hung in strings from the coacervate of high arabinate percentage (the vacualization is not indicated in the figure). This condition is seen in the early stages of complex coacervation when the celloidin membrane has been prepared from an emulsion of the colloid stock solution in the ether-amyl alcoholic celloidin solution, which has been left at room temperature for some considerable time. The emulsified drops of the colloid stock solution have then apparently passed into a condition of gelation. Such pictures in which a central mass is connected with the wall by strings has been previously described for the complex coacervate gelatine + gum arabic and there the conditions for the formation were identical. The only difference is that here the central mass consists of one large cohering mass of the coacervation of high nucleinate percentage or of a greater number of smaller masses packed closely together, which becomes apparent on staining with methylgreen [1]).

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[1]) Gelatine for bacteriological purposes of the "Lijm- en gelatinefabriek Delft" at Delft.

[2]) Gomme senegal, petite boule blanche I" of the firm of Allan et Robert, Paris.

[3]) Na-nukleinat aus Hefe, of Schering-Kahlbaum, Berlin.

[4]) H. G. BUNGENBERG DE JONG, B. KOK and D. R. KREGER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 512 (1940).

[5]) H. G. BUNGENBERG DE JONG and B. KOK, loc. cit.

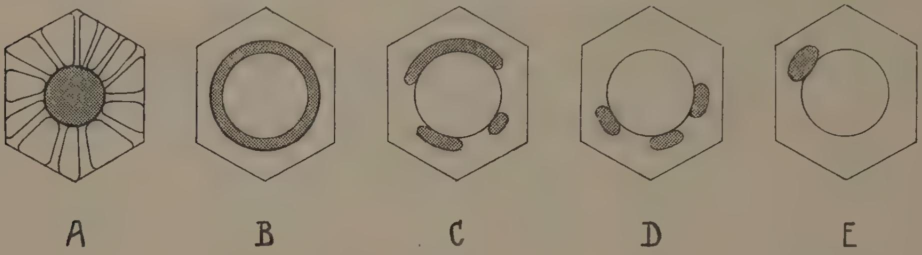

Fig. 1.

The formation of strings is the outward sign that the coacervate is not, or rather, not yet in a very liquid condition, but in a plastic intermediate stage between liquidity and solid gelation.

Therefore the picture does not remain like this with the temperature employed; at least the coacervate of high arabinate percentage becomes very liquid, whereas the coacervate of high nuclein percentage becomes liquid, but remains highly viscous. In course of time the other pictures of fig. 1 may arise, which otherwise will also arise when the emulsion used in casting the membranes has been left at room temperature during a shorter period of time.

After a confusion of forming coacervate drops those of the complex coacervate of high arabinate percentage soon coalsce to one coacervate which in course of time becomes free from the vacuoles first enclosed. For the location of the coacervate of high nucleinate percentage (grey in the figure) we then find the pictures of fig. 1 B—E, that is to say it is always located against the central vacuole. From the discussion in 5 c it may be concluded that the condition of fig. 1 B, in which the central vacuole seems quite surrounded by a layer of the coacervate of high nucleinate percentage is only a picture which may occur incidentally when the preparation is sharply focussed at a certain depth.

4. *Changes in course of time.*

The pictures of fig. 1 B—E are changed gradually in course of time, because nucleic acid is continually removed from the cell. As coëxisting coacervates in the gelatine — gum arabic — nucleinate system at a given pH are only possible in a certain section of mixing proportions of the three colloids, the relative mixability of the coëxisting coacervates increases on continued withdrawal of one of the three colloids (here nucleinate) and when too much nucleinate has been drained off coëxisting coacervates are no longer possible. Before this occurs the coacervate of high nucleinate percentage becomes richer in water, which may find expression in the microscopic picture as a (not very striking) increase of volume of the coacervate of high nucleinate percentage. Soon however the decrease of volume of this coacervate preponderates in consequence of the increased mutual solubility of the two coacervates.

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[1]) In a medium of only diluted acetic acid methylgreen strongly stains the coacervate of high nuclein percentage, whereas the coëxisting coacervate of high arabinate percentage is weakly or not at all stained. When to the buffer used here we add a little methylgreen the coacervate of high nuclein percentage is very weakly stained, the coacervate of high arabinate percentage is not stained. The great decrease of the intensity of the staining is due to the presence of a salt (Na acetate) in the medium liquid.

Compare communication II.

So when the medium liquid continues to be conducted past the membrane we see that gradually the volume of the coacervate of high nucleinate percentage decreases and finally disappears altogether.

When the coacervate of high nucleinate percentage is at first divided into several cohering masses (fig. 1 C and D) these do not disappear simultaneously, but the mass which was originally largest continues longest, so that the condition of fig. 1 E, — in which there is one coacervate drop of high nucleinate percentage on the boundary of the coacervate of high arabinate percentage and the large central vacuole — is always passed.

5. *Discussion.*

a. *What will be the final condition when the nucleinate cannot be removed?*

In 4. we have seen that pictures as in fig. 1 E are always passed on dissolution of the complex coacervate of high nucleinate percentage which is divided into several distinct masses.

The condition pictured in fig. 1 E however acquires a more fundamental significance in view of the following considerations. When we suppose that it is experimentally possible to make the celloidin membrane sufficiently impermeable, so that no nucleinate is lost, it is to be expected that after sufficient duration a condition will yet arise as in fig. 1 E. The coacervate of high nucleinate percentage divided into several cohering masses in fig. 1 C and D possesses greater boundary face energy than if it had coalesced to one cohering mass.

With the comparatively low temperature (ca. 33°) at which we have worked the coacervations are however so viscous, that convection currents in the cells become extremely slight so that the separate masses have no opportunity of coming into contact and coalescing.

b. *Morphological model of the plant cell.*

The morphological picture of fig. 1 E, which as we have seen in *a* would be the final condition if the celloidin membrane was not permeable to nucleinate is much like that of a mature plant cell. There too a cohering body rich in colloids — the nucleus — is embedded in a liquid rich in colloids — the cytoplasm — which encloses another liquid poor in colloids — the central vacuole.

c. *The location of the coacervate of high nucleinate percentage against the central vacuole.*

As regards one detail however we are not sure if the analogy discussed in *b* is also applicable here, namely the question of the complete embedding of the nucleus in the cytoplasm, i.e. that the nucleus though it may be pressed against the vacuole, must yet be separated from the vacuole liquid by a thin layer of cytoplasm, even if this layer is so thin as to be invisible. We have not yet been able to observe such a covering of the coacervate of high nucleinate percentage by a visible film of the coacervate of high arabinate percentage. So there are here two possibilities:

a. There is here nevertheless an invisible layer of coacervate of high arabinate percentage (I), between the coacervate of high nucleinate percentage (II) and the equilibrium liquid (see scheme fig. 2 A),

b. part of the surface of the coacervate of high nucleinate percentage lies immediately against the equilibrium liquid (see scheme fig. 2 B).

This alternative means that either there is "three phase contact" (B) between the two coëxisting coacervates and the equilibrium liquid, or there is not (A). But complications may have arisen owing to the facts that we made the experiments in the celloidin membrane at a comparatively low temperature, and that in consequence of their approaching

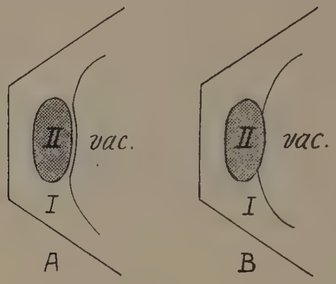

Fig. 2.

a condition of gelation the coacervates at least the coacervate of high nucleinate percentage was in a highly viscous intermediate stage.

We have therefore verified if an unmistakable answer to the alternative may be obtained at higher temperature and apart from possible complications of enclosure in the celloidin membrane.

When at 40° coacervation is caused by adding to 10 cc of a stock solution of colloids $(3\,G + 1\,A + 1\,N + 100\,H_2O)$ 20 cc of the in 3) mentioned buffer and when the composite drops are viewed through the microscope on a starched object glass at 40°, the pictures of these drops are usually as shown in fig. 3 A the coacervate of high nucleinate percentage is surrounded on all sides by the coacervate of high arabinate percentage. But when the coacervated system is seen in a cuvette through a microscope turned horizontally, it appears that again the coacervate of high nucleinate percentage is pressed against the boundary of the coacervate of high arabinate percentage and the surrounding equilibrium liquid (fig. 3 B). Owing to the fact that the coacervate of high nucleinate percentage has greater specific gravity than the coacervate of high arabinate percentage, the centre of gravity of the composite drop will be below the point of application of the upward pressure and so the composite drops viewed horizontally must take up the

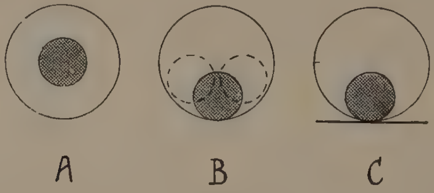

Fig. 3.

position of fig. 3 B in the equilibrium liquid (or at least they will have to fluctuate round the position of fig. 3 B owing to convection currents in the equilibrium liquid). When such composite drops sink from the surface of a starched object glass (starching prevents the coacervate from running over the glass surface), they will lie on it as in fig. 3 C, which corresponds with the picture of fig. 3 A when the drops are viewed vertically.

From the above it is clear that also apart from possible complications owing to the celloidin membrane and at temperature at which the coacervates are sufficiently liquid, the location of the two coëxisting coacervates and of the equilibrium liquid will be again such that we are confronted with the same question; is there "three phase contact" or does an extremely thin layer (or film) belonging to the coacervate of high arabinate percentage yet separate the coacervate of high nucleinate percentage from the equilibrium liquid. We do not dispose of further data which enable us to answer this question either positively or negatively.

Summary.

1. A suitable selected mixture of gelatine, gum arabic and Na-yeast nucleinate sols is enclosed in the cells of a celloidin membrane and coacervation is caused by conducting past the membrane a liquid of suitable pH.
2. The positions with regard to each other of the two demixing coëxisting complex coacervates and the equilibrium liquid is observed. In principle we find here analogy with the morphology of the mature plant cell. The coacervate of high arabinate percentage (analogous with the cytoplasm) becomes parietal and surrounds the equilibrium liquid (analogous with the central vacuole) while the coacervate of high nucleinate percentage (analogous with the nucleus) is embedded in the coacervate of high arabinate percentage.
3. The coacervate of high nucleinate percentage is pressed against the boundary of equilibrium liquid and coacervate of high arabinate percentage, as more or less rounded bodies. It is uncertain whether there is here "three phase contact".
4. Owing to the fact that the celloidin membrane is not sufficiently impermeable and gradually allows nucleinate to diffuse, the coacervate of high nucleinate percentage in course of time disappears.

Leiden, Laboratory for Medical Chemistry.

Geology. — *Contribution to the petrography of Bintan (Riouw-Lingga Archipelago).* By J. J. HERMES and D. R. DE VLETTER. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of December 27, 1941.)

The Billiton Mij. donated to the "Geologisch-Mineralogisch Instituut" of Utrecht, Holland, a collection of rocks and the reports of Dr. P. M. ROGGEVEEN, who, in 1930, made geological investigations on several islands of the Riouw-Lingga Archipelago. The material from Bintan and surrounding islands has been examined by the authors.

Historic review.

EVERWIJN (lit. 3) already noticed that plutonic rocks seem to dominate in the N part of Bintan. The mountain Bt. Bintan Besar consists of a fine grained diorite, while several hills along the coast and in the inland generally are built up by very coarse grained granite. Iron ore is widespread; it occurs partly as a fine-grained magnetical sand in all the valleys, partly in large quantities as brown iron ore.

BOTHÉ (lit. 1, 2) states that a biotite granite, sometimes containing amphibole occupies important parts of the island. From the Bt. Batoe Besar and G. Lengkoeas he mentions alcaligraniteporphyry, and from Bt. Bintan Besar diorite, which he considers to be differentiates of the granitic magma. GISOLF (in BOTHÉ 2) is of the opinion that the granite of Bintan forms the top of a batholith. LOTH (in BOTHÉ 2) distinguishes the Batam and Bintan type of granite, the former, from the western part, having flesh-coloured orthoclases and greenish plagioclases, the latter colourless felspars. BOTHÉ mentions diabase close to the granite on P. Boeau. Furthermore liparite (G. Kidjang) and black porphyries (G. Bintan Ketjil) occur, whose connection with the granite is still obscure. If they correspond with the eruptive rocks from the Pahang Volcanic Series, they are older than the granite. On the other hand quarzporphyries from Batam (Tering Bay) are considered to be younger than the granite, because they contain sharp-edged granite inclusions. BOTHÉ found pneumatolytic phenomena (tourmaline-greisen, probably with cassiterite) in the granite of the N coast of Bintan, P. Soempat, P. Ngiri and P. Ranggas. Unweathered sedimentary rocks are only exposed on a small scale, the greater part of Bintan being covered by rocks of bauxitic and limonitic composition. Fossils have not been found. The sediments belong to two formations. The older one is strongly folded and is built up by dynamometamorphic arenaceous and argillaceous rocks: phyllites, metamorphic liparite- and dacite-tuffs and quartz schists which are regarded by WESTERVELD (5) as triassic. According to BOTHÉ they have been intruded by the granite. The younger formation, which occurs only in the S and SW, is much less folded and consists of clay shales and sandstones with streaks of coal. They lie disconformably upon the older formation (Tg. Enim). BOTHÉ (2) states that the younger formation resembles the tertiary "Plateauzandsteen" of W Borneo.

ROGGEVEEN, in his report, mentions the existence of numerous inclusions in the granite between Tg. Tondang and Tg. Said, on P. Mantjin and on P. Noembing. At Tg. Gading and on P. Noembing dark granite is intruded by a lighter one. In contradiction with BOTHÉ's opinion, pneumatolysis is not important in the granite massive of N. Bintan, where, W of Pengoedang tourmaline is observed in the granite, nor is the NW part of Bintan rich in greisen. In SW Lobam the granite is traversed by lamprophyric dykes. The granites from Lengkoeas, N Boeton and Poto show transitions to quartzporphyritic rocks. BOTHÉ's opinion that rocks of Lengkoeas are alcalic is unestablished, as alcalic minerals do not occur. Only chemical analysis might confirm his opinion. Pneumatolytic phenomena do not occur in the granites of Lengkoeas, Boeton and Poto. Noembing and surrounding islands are built up exclusively by granite. On the E coast of Telang a black quartzporphyritic rock has been found. Contactmetamorphic sediments of Pengoedang prove the granite to be younger than the sediments; there are no samples of this locality in our collection. ROGGEVEEN states that black quartzporphyries occupy the principal hills of Bintan (Bt. Bintan Besar, Bt. Bintan Ketjil and G. Kidjang); he did not find the diorite mentioned by former authors from Bt. Bintan Besar. Furthermore, black quartzporphyries are mentioned from the SW part of P. Mantang, E part of Telang Ketjil, SE part of Boeton and P. Mepoeroe. On the last two islands they occur together with granite. On P. Ranggas (S Bintan) strongly tourmalinized rocks occur. They might be altered quartzporphyries. ROGGEVEEN states that the geologic appearance of the quartzporphyries indicates a genetic connection with the granites. They form the highest tops in regions where the granite appears on a lower level. The granite shows transitions to the quartzporphyries which have a much more acid composition than the permocarboniferous porphyries of W. Batam and environs. Permocarboniferous sediments have not been found. This leads to the conclusion that the black quartzporphyries are probably the quickly cooled top of the granite massive. They may correspond with the black quarzporphyry of the Tering Bay (Batam) which, moreover, has the same habit.

ROGGEVEEN mentions slightly folded sand- and claystones from the SW part of Bintan and from the E part of P. Dompak; they contain thin coallayers with unrecognizable vegetable material. These rocks correspond with rocks from N Batam. On P. Seraja the same formation presents conglomeratic intercalations; the pebbles consist of arenaceous and siliceous material. It is a pity that these pebbles do not occur in our collection. On P. Los, Pangkil and S of Tg. Sebong white and grey-white sandstones were found. A formation with quite another habit is mentioned from P. Sekiri. It consists of argillaceous sandstones and sericiteshales. These rocks make the impression to be altered effusiva. This formation shows a great resemblance to the permocarboniferous of W Batam. ROGGEVEEN supposes that all the sand- and claystones of Bintan belong to one formation because the contactmetamorphics S of Pengoedang show low dips. This argument appears to us rather poor as BOTHÉ mentions the existence of an angular disconformity from one place in the S, whilst from other localities in the N high dips were reported.

Description of Rocks.

Among the abyssal rocks we can distinguish granitic, granodioritic and syenitic ones. We shall describe them separately.

The *granitic* type is represented by the following samples: 41, 42, 45, 46 from P. Noembing, 92, 94, 96, 101, 102, 105, 107, 108, 111 from P. Telang, 267, 271 from P. Mantjin, 515, 516 from Tg. Gading, 517, 518, 519, 520 from Tg. Tondang, 544, 545 from Tg. Bintan, 548, 549 from P. Ranggas (N Bintan), 551 from P. Marawang, 552, 553 from Tg. Said, 732 from P. Dompak. They are generally fresh, phanerocrystalline, leucocratic rocks, consisting of partly pink and green, partly grey to green felspars, quartz and biotite. Amphibole and muscovite may occur but biotite is the leading femic mineral. The slides consist chiefly of quartz and perthitic, sometimes sericitized orthoclase. Albite-oligoclase, always present, often shows twin lamelling and occasionally zoning structures. Biotite, sometimes bleached, is found in modest quantity, with zircon in pleochroitic haloes. Amphibole and muscovite occur in a few cases. Accessories are zircon, apatite and magnetite. The femic minerals are sometimes changed into an aggregate of chlorite and epidote. Fluorite and cassiterite give witness of pneumatolytic action. Generally the texture is hypidiomorphic, though, in 108, 266, 516, 517, 544, graphic textures were observed. Cataclastic phenomena are not rare: undulatory extinction of quartz, bent mica flakes and, in 105, a mylonitic zone. Summarizing we have mostly leucocratic biotite-granites, amphibole-biotite-granites and muscovite-biotite-granites. The two types of granite as described by BOTHÉ could be distinguished in our collection. It must be observed that 732 has been collected on P. Dompak, where ROGGEVEEN only indicates "sediment". Possibly sample 732 is a pebble from a conglomerate.

Grano-diorites: 86, 88, 97, 98, 99, 103, 104, 106 from P. Telang, 543 from Tg Bintan, 546 from S of Pengoedang, 557 from P. Soempat, 573, 574, 582 from P. Lobam. They are fresh, phanerocrystalline rocks, consisting of quartz and grey to green felspars. The slides are chiefly composed of quartz, mostly twinned albite-oligoclase, often sericitized, and of varying quantities of sometimes perthitic, sometimes sericitized orthoclase. Biotite and less amphibole are the dark constituents. Accessories are zircon, apatite, magnetite and leucoxene. Fluorite and cassiterite bear witness of pneumatolytic action. Beautiful graphic textures are widespread: sometimes the whole rock is micropegmatitic. We can subdivide the granodiorites into two transitional groups. The first group, occurring on P. Telang, generally has a propylitic habit, the femic constituents being converted into a mass of chlorite and epidote, the felspars getting dull. It is noticeable that the more important the graphic textures are, the more important is the change into chlorite and epidote. The second group, occurring in the N of Bintan, has a much fresher habit; chlorite and epidote are rare; graphic textures are not frequent. Cataclastic phenomena, as undulatory extinction of quartz and bent mica plates are not rare. A very remarkable feature was represented by 575, from Selat Kidjang, a granophyric rock, which has been completely replaced by hydrargillite. (Plate, fig. 3, 4.)

The *syenites* 266, 268, 269 and 270 are all from P. Mantjin, where they occur together with granite (267, 271). They are phanerocrystalline rocks, consisting of yellowish grey to brownish red felspars and dark-green amphibole. In the slides the principal constituent is a sometimes perthitic orthoclase. The amphibole is green, with unusually strong pleochroism from yellowish brown and green to very dark green. Anorthoclase occurs in a few grains, which show twin lamelling. Filling the spaces between the other constituents, albite-oligoclase occurs. The albite must have been crystallized after the orthoclase and sometimes apparently on the expense of it. This might point to hydrothermal origin. Nepheline and quartz are absent. Some dark green mica's occur. A rather great amount of pyrite is found. Accessories are apatite and zircon.

General review and details about contacts. The granites and granodiorites merge into each other; the granodiorites are poorer in orthoclase and show more graphic intergrowths. Clearly, these groups belong together. The occurrence of granites and syenites on P. Mantjin seems to prove that also the syenites are differentiates from a granitic magma. 557 from P. Soempat, N of Bintan shows a contact between a granodiorite and a quartzporphyrite: the first seems to be an inclusion in the second which, on the other hand, is metamorphosed, as minute secondary amphibole crystals, filling corrosion-cavities of quartz, occur. 111, from P. Telang is again a contact between a granite and a quartz-porphyrite, which contains a grain of tourmaline. 44, from P. Noembing, is a granite pegmatite with inclusions of granodioritic material. 46, from the same island is a granite with granodioritic parts. 94 from P. Telang shows a contact of a granodiorite merging into a rock composed of quartz, epidote, some garnet and large nests of magnetite.

Aplitic and pegmatitic dykes. 45, 48, 49 from P. Noembing and 89, 91, 109 from P. Telang are aplites. With the exception of 91, they show transitions into aplitic granites. They are grey-white to pink rocks with quartz, sometimes perthitic orthoclase and albite-oligoclase. The felspars are slightly sericitized. Accessories are biotite, zircon and magnetite. The occurrence of fluorite proves pneumatolytic action. Graphic textures are common. In 109 the aplite is an inclusion in a malchitic rock. In 91, a granodiorite aplite, albite-oligoclase is predominating. In this sample a fair amount of fluorite occurs.

Only two samples of granite pegmatite were at our disposal, 43 and 44, both from P. Noembing. They are coarse-grained rocks with grey-green felspar, quartz and biotite. The slides consist of quartz, perthitic orthoclase and some albite-oligoclase. The felspars are sometimes sericitized. Biotite, partly chloritized, occurs rather rarely. Accessories are apatite, zircon and magnetite. Some fluorite and pyrite occur. 43 shows graphic texture. 44 contains inclusions of granodiorite.

Graniteporphyry. 550, from P. Ranggas (N Bintan), is a light coloured rock with phenocrysts of quartz and felspar. The groundmass consists of orthoclase, albite-oligoclase and quartz. The texture is microgranitic. Phenocrysts of quartz, idiomorphic and corroded, of perthitic orthoclase, albite-oligoclase and biotite occur.

Granodioriteporphyrite. 90, from P. Telang, is a grey-pink rock with phenocrysts of quartz and felspar. The groundmass consists chiefly of lath-shaped felspars and quartz, showing micropegmatical intergrowth. Phenocrysts of felspar are albite-andesine with inclusions of epidote. Fluorite occurs locally. 579, 580 and 581 from G. Koe are transitional between granodioriteporphyrite and quartzporphyrite. Macroscopically they strongly resemble 90. Microscopically their groundmass appears to be finer grained. Beside quartz and albite-andesine, biotite phenocrysts, partly altered into magnetite and limonite occur. In 579 epidote and zoisite are found. Biotite is also found in the groundmass.

Rocks of "malchitic" composition from N Bintan. 518, 519, 552, 553, 554, 555. They are dark, greenish rocks, in which we can distinguish felspar, biotite and amphibole in about isometric grains. The slides consist of twinned oligoclase-andesine laths, sometimes more acid, of partly chloritized biotite and of less green amphibole. Quartz is accessorial. Where it occurs, it fills the spaces between the other minerals, in that case showing simultaneous extinction over large areas. Zircon, apatite and magnetite are accessorial. All these rocks are in some way connected with granitic rocks. 518 clearly shows the "malchite" to be an inclusion in the granite. As to the other rocks, we could not establish the connection with certainty; in one sample, however, inclusions of pegmatitic quartzdiorite occur within the "malchite". ROGGEVEEN mentions from the same area inclusions in the granite, rich in mica, which he considers to be resorbed parts of the roof of the batholith. We assume our rocks to be identical with these inclusions. In 552 in one place a very large quartz crystal is found, much larger than the phenocrysts of the "malchite". It is rounded and shows a reaction rim, formed by concentrations of amphibole and biotite, accompanied by apatite and titanite. The quartz contains some inclusions of irregular, sericitized felspar, which locally seems to have penetrated the quartz. 738 is a "malchite" from P. Dompak, which shows the same composition as the rocks described above. ROGGEVEEN in his map, indicates on P. Dompak only sediments and at locality 738 a conglomeratic sandstone. Possibly our sample is an element of this conglomerate, although its habit does not give the impression (compare granite 732).

The *malchite* 572 from P. Lobam differs from the foregoing samples. It is a dense, blue-black rock, which consists of weathered oligoclase-andesine laths, abundant amphibole, from small crystals to phenocrysts and some quartz. Zircon and many magnetite grains are accessorial. It differs from the foregoing in the content of quartz, the absence of biotite, the more weathered state and the more porphyritic texture. Moreover, it is treated apart as ROGGEVEEN indeed mentions lamprophyric dykes from P. Lobam, so, in this case we have to do with a real malchite. 81, 82, 84, 85, 87, malchites, from P. Telang are fine-grained, dark, blue-black rocks, which consist microscopically of a very fine-grained groundmass of quartz and abundant amphibole in small idiomorphic prisms and in irregular masses, and of albite-oligoclase. In this groundmass we find phenocrysts of quartz, ? albite-oligoclase and amphibole. The felspars are thoroughly silicified and kaolinized. A few grains of apatite occur. In one case (82), secondary quartz occurs in idiomorphic bipyramids. In 87 needleshaped amphibole and albite are found along joints. 84 shows a concentration of zoisite. With 84 a problem arises as to the nature of these malchites. Here we find angular very fine grained parts, practically without amphibole. We got the impression that we have to do with a tuff breccia in which the amphibole might have originated by contact metamorphism. On the other hand the facts that we find in five samples the amphibole in regular distribution and that 95, a quartzporphyrite, also from P. Telang, contains a malchitic vein, support our opinion that we have to do with true malchites. ROGGEVEEN's report only indicates black porphyries from this place.

Three samples of *greisens* (83, 100, 110) from P. Telang were found to be tourmaline greisens. They consist of abundant quartz, tourmaline and a colourless, spherulitic mica in varying proportions, a few grains of cassiterite and accessorial zircon. Topaz was not found. The transition from granite to greisen could be observed.

A *diabase*, (50) from P. Boeton is a blue-black aphanatic rock, which chiefly consists of sericitized albite laths. Here and there twinning was observed. The space between the felspars is occupied by tiny sericite flakes and ore grains. Accessories are zircon and abundant apatite. The rock might be a sericitized and albitized diabase.

Some samples consist principally of *ore*. 36 and 38 from P. Ranggas (S. Bintan) are manganite which, metasomatically, has replaced a quartzporphyrite. 735 from P. Dompak is magnetite with some limonite. 35 from P. Ranggas (S. Bintan) and 570 from P. Soempat are limonite.

558, 565, 567 from P. Soempat and 93 and 94 from P. Telang are *epidote rocks*. They consist chiefly of epidote, partly in idiomorphic crystals with good cleavage, partly in aggregates, and of some quartz, which occupies the space between the epidote. 93 contains beside epidote and quartz rather much fluorite.

Quartzporphyries. 261 from P. Mantang and 547 from P. Ngiri. The first is a black rock with some phenocrysts, the second is light green (epidotisation). In the somewhat devitrified fluidal groundmass of 261 many small quartzsplinters, and corroded quartz- and felsparphenocrysts occur. The latter are partly sericitized or altered into epidote and chlorite. Perthitical orthoclase dominates among the felspars. Probably part of the epidote and chlorite is pseudomorphic after biotite. Some magnetite and titanite were observed. 547 is epidotized on a greater scale. The very clear and corroded quartzphenocrysts often are cataclastic and show undulatory extinction. The orthoclase and acid plagioclase phenocrysts are dusty. Epidote and zoisite are abundant.

Quartzporphyri(t)es. 29, 30, 31, 32, 33, 36, 37, 39, 40 from P. Ranggas (S. Bintan), 583, 584, 585 from Selat Kidjang, 713, 715, 718, 721, 722, 723, 724 from P. Sikiri, 714, 717, 719, 720, 725, 726, 727 from Bt. Manok. These, generally grey-white to pink rocks are very much altered. Microscopically the felspars appear to be altered to such a scale, that they can not be definitely determined. The less altered 31 and 32 have a groundmass of quartz, sericite and kaolin; the phenocrysts are corroded quartz and felspars, altered into sericite concentrations. Veinlets of secondary quartz indicate the beginning of silicification. This process is already advanced in 713, 715 and 721, the groundmass being silicified, while in 713 quartz is probably pseudomorphic after felspar. Sericitization is also found in 722, 723, 726. In some samples from P. Ranggas, tourmalinization occurs. In 33 the tourmaline appears in veins and in irregular masses; in 29, 30, 37 tourmalinization has advanced, and here even the felsparphenocrysts are partly tourmalinized. The tourmaline occasionally displays spherulitic textures. In 30 we observed idiomorphic quartz in a tourmaline veinlet. In 39 and 40 the tourmaline is clearly younger than the quartz, the latter being often surrounded by tourmaline aggregations. Many of these pink-coloured rocks show hydrargillitization, often accompanied by limonitization. Microscopically we see in 583, 584, 585 and 586 clear, corroded quartz and sometimes quartzsplinters in a hydrargillitized and limonitized groundmass. Beautiful lamelled hydrargillite often occurs in the quartz along many fissures and in corrosion-cavities. These remarkable features strongly point to hydrargillitization of the quartz (Pl., fig. 1, 2). Limonite concentrations, in 584 containing a felspar-relic, are pseudomorphic after felspar. 714, 717, 718, 719, 721, 722, 725, 727 and 731 are hydrargillitized quartzporphyri(t)es which contain less limonite than previous samples; felspars, here, have also been altered into hydrargillite.

Quartzporphyrites. 51 from P. Boeton, 262, 272, 274, 275 from Bt. Bintan Ketjil, 559, 560, 561, 562, 563, 564, 568, 569, from P. Soempat, 736 from P. Dompak. They are black and light-grey rocks. The groundmass is generally fine grained. The felsparphenocrysts (albite to andesine, partly replaced by epidote, chlorite and even amphibole) are sometimes sericitized, the quartzphenocrysts often corroded. Some orthoclase may occur. In 262, the groundmass is partly vitreous; fluidal texture occurs. The groundmass of the metamorphosed quartzporphyrites of P. Soempat (559, 560, 561, 562, 563, 568, 569) always contains epidote, sometimes biotite and often amphibole, which occurs also in large crystal-skeletons. Occasionally silicification occurs. In 569 amphibole occurs in veinlets; epidote and zoisite in large crystals. In 560 and 562 they form veinlets; 563 contains some garnet. In 562 the quartzporphyrite is in contact with a dark rock, chiefly composed of brown and blue-greenish amphibole, epidote and zoisite. Approaching the contact the coloured minerals of the quartzporphyrite gain in importance. 564 is an albite rock, probably an altered quartzporphyrite, penetrated by an amphibole veinlet. 272, 274 and 275 are equally metamorphosed rocks rich in amphibole, in the groundmass arranged in streaks, while large crystals often occur together with epidote, titanite or biotite. They give the impression of having replaced felsparphenocrysts. Other ones have been replaced by sericite, calcite and epidote. Epidote and biotite, moreover, occur in the groundmass. 272 contains an inclusion of a silicified black porphyrite with veinlets, consisting of amphibole, epidote, quartz, biotite and felspar. 274 contains an inclusion of a fine grained quartzporphyrite with a vein, filled nearly exclusively with amphibole at the outside and epidote-zoisite in the centre.

Quartzporphyritical tuffs. 576, 577, 578 from P. Mepoeroe. They are metamorphosed rocks, containing components, which in their groundmass all show secondary biotite and amphibole, partly in crystal-skeletons, often arranged in streaks. Phenocrysts of quartz (strongly undulatory) and felspar are common. One component of the tuffs containing less phenocrysts, shows fine fluidal textures, another component consists of an intergrowth of acid felspar laths with amphibole.

Porphyrites. 263 and 264 from Bt. Bintan Besar are black porphyritic rocks, with devitrified and locally fluidal groundmass. Felspar (albite to andesine, often cataclastic, sericitized and sometimes with calcite) and amphibole (sometimes with titanite and epidote) are the phenocrysts. Magnetite, apatite, zircon and chlorite are accessories.

Hydrargillite-limonite rocks ("Bauxites"). Our collection contains different hydrargillite-limoniterocks, the original nature of which could not be stated: 730 from P. Dompak, 733 from the S of Pengoedang, 734 from Sengarang, 716 from P. Sikiri.

Sediments: Only five samples of sediments were available, all being of an arenaceous nature. It is a pity that so few samples have been collected. Without doubt this is caused by the fact that ROGGEVEEN's exploration was in the first place a prospection for tin; also the fact that exposures are rare has certainly influenced the number of samples. 729 from P. Dompak and 739 from an unknown locality are quartzitic arkoses, consisting of rounded, isometrical quartz grains, being partly cemented by silica. The rocks contain many rounded, brownish grains (probably ex-glauconite). 729 contains an inclusion of black shale and 739 of vegetable material. Some sericite concentrations (? ex-felspar) occur. Accessories are zircon and tourmaline. 276 from Bt. Bintan Ketjil is a quartz-sandstone made up of isometrical quartz grains, with a single prism of tourmaline, secondary biotite and possibly some amphibole. 556 from NW Bintan is a quartz-mylonite, consisting of great, strongly undulatory quartzes. In some places crushing has produced mosaic texture. The cement between the quartzes is locally triturated to a great extent. Some tourmaline grains occur. 728 from G. Manok (W Bintan) is a sericite quartzite, composed of isometric quartz grains and sericite. Secondary tourmaline (yellow to brown) occurs in many small prisms.

SUMMARY.

In connection with the facts, that samples are relatively scarce and the field notes of ROGGEVEEN were not available, a definite account of the age relations of the different rocks can not be given. We will review the petrographic units separately, at the end trying to establish their mutual relations. The intrusive rocks consist of granites and granodiorites, with their dykes, and syenites. The first two groups belong together, as they gradually merge into each other: epidotization, chloritization and graphic textures increasing to the basic side. They are accompanied by pegmatitic and aplitic dykes, and in one case by a malchite (P. Lobam). A granite porphyry from P. Ranggas (N Bintan) and a granodiorite porphyrite from P. Telang probably belong to the granites and granodiorites. The occurrence of syenites on P. Mantjin is remarkable, syenites being very rare in the Netherlands East Indies. As they occur together with granites, they have probably been derived from the same magma. From N Bintan a couple of rocks of "malchitic" composition were examined, which were identified with the dark inclusions in the granite mentioned by ROGGEVEEN. On P. Telang malchitic rocks of an uncertain nature are found. A veinlet of malchite in a quartzporphyrite points to true dike rocks. On the other hand one of the malchites was built up by different components, thus pointing to a metamorphosed quartzporphyritic tuff. In one case a contact of a granite with a quartz-epidote-garnet rock was met with (P. Telang). An albitized and sericitized diabase has been sampled from P. Boeton. The effusive rocks show many varieties. We found quartzporphyries, quartzporphyrites and porphyrites. Occasionally they are strongly altered. Beside amphibolization, tourmalinization, epidotization and chloritization, also hydrargillization was observed. Some rocks, which consisted exclusively of hydrargillite, limonite and some quartz ("Bauxites") are probably thoroughly altered quartzporphyri(t)es. With regard to the hydrargillitization two remarkable facts have been found. Firstly hydrargillite-bearing solutions have apparently changed the quartz of quartzporphyri(t)es partly into hydrargillite (Pl., fig. 1, 2). Secondly the granophyric rock 575 has been completely changed into hydrargillite, the original nature of the rock being proved by its textural features (Pl., fig. 3, 4). Pneumatolytic phenomena were often met with in our material: fluorite was present in the intrusiva, three tourmaline greisens were found, and quartzporphyri(t)es from P. Ranggas show advanced tourmalinization. As to the connection between the granites and effusive rocks, facts seem to be contradictory. 1. On P. Soempat a quartzporphyrite, 557, contains an inclusion of granodiorite. On the other hand the quartzporphyrite was metamorphosed, which was shown by abundant small secondary amphiboles. 2. Quartzporphyri(t)es from P. Ranggas show advanced tourmalinization. 3. Amphibolized quartzporphyrites occur on Bt. Bintan Ketjil. 4. On P. Mepoeroe amphibolized quartzporphyritical tuffs were found. These facts lead to the following assumption. The quartzporphyrites contain homogeneous inclusions of an older granite and were afterwards metamorphosed. It is a pity that only five sediments were sampled. They were all of an arenaceous character, varying from quartzitic sandstones and arkoses to, in one case, a sericite quartzite with tourmaline. All contained some tourmaline. The relations between sediments and abyssal rocks still remain uncertain. According to our opinion the sediments may belong to two formations, two samples 728 and 276 are contactmetamorphic; they contain secondary tourmaline; 728 moreover secondary sericite and 276 secondary biotite and possibly amphibole. Two of the other samples, 729, 739 are rocks, which have originated from glauconitic arkoses. They have quite another habit than the foregoing samples. It is uncertain whether they are younger or older than the granite of our region; the tourmaline prisms of 729 may be authigeneous or clastic. Their value can only be judged at a time when ROGGEVEEN, now away from Europe, will be in the possibility to give more field evidence.

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J. J. HERMES and D. R. DE VLETTER: Contribution to the petrography of Bintan (Riouw-Lingga Archipelago).

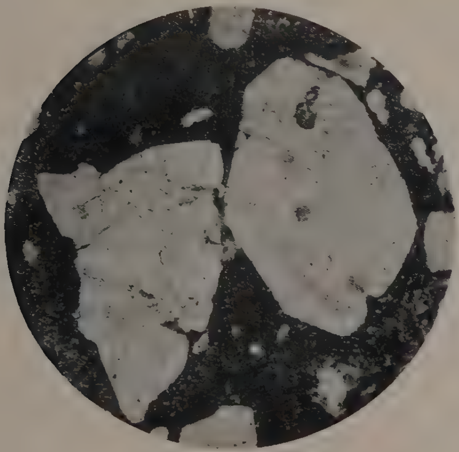


Fig. 1. 20 ×
Parallel Nicols.

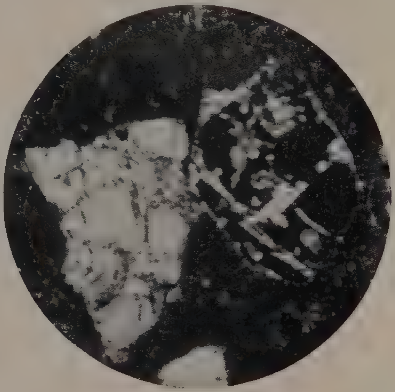


Fig. 2. 17 ×
Crossed Nicols.

Fig. 1, 2. Quartzporphyr(ite); no. 584, Selat Kidjang, S. Bintang.
(Phenocrysts of quartz with veinlets of hydrargyllite).

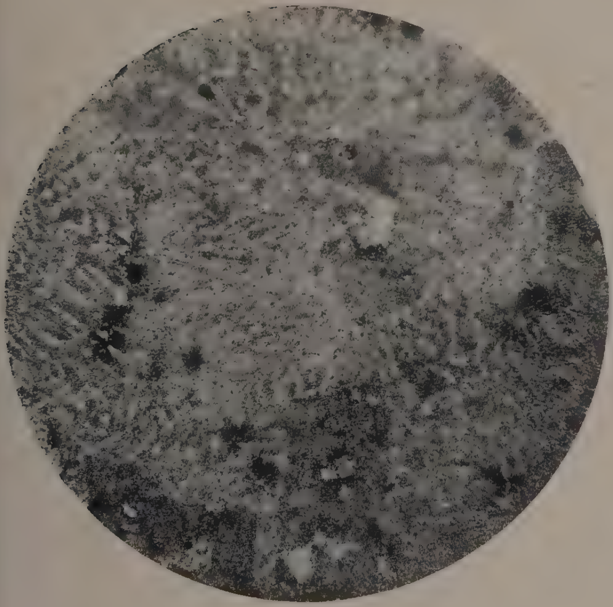


Fig. 3. 90 ×
Parallel Nicols.

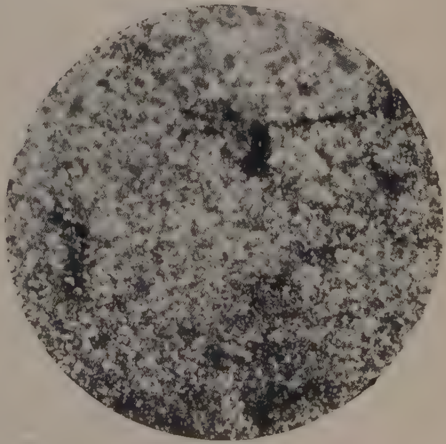
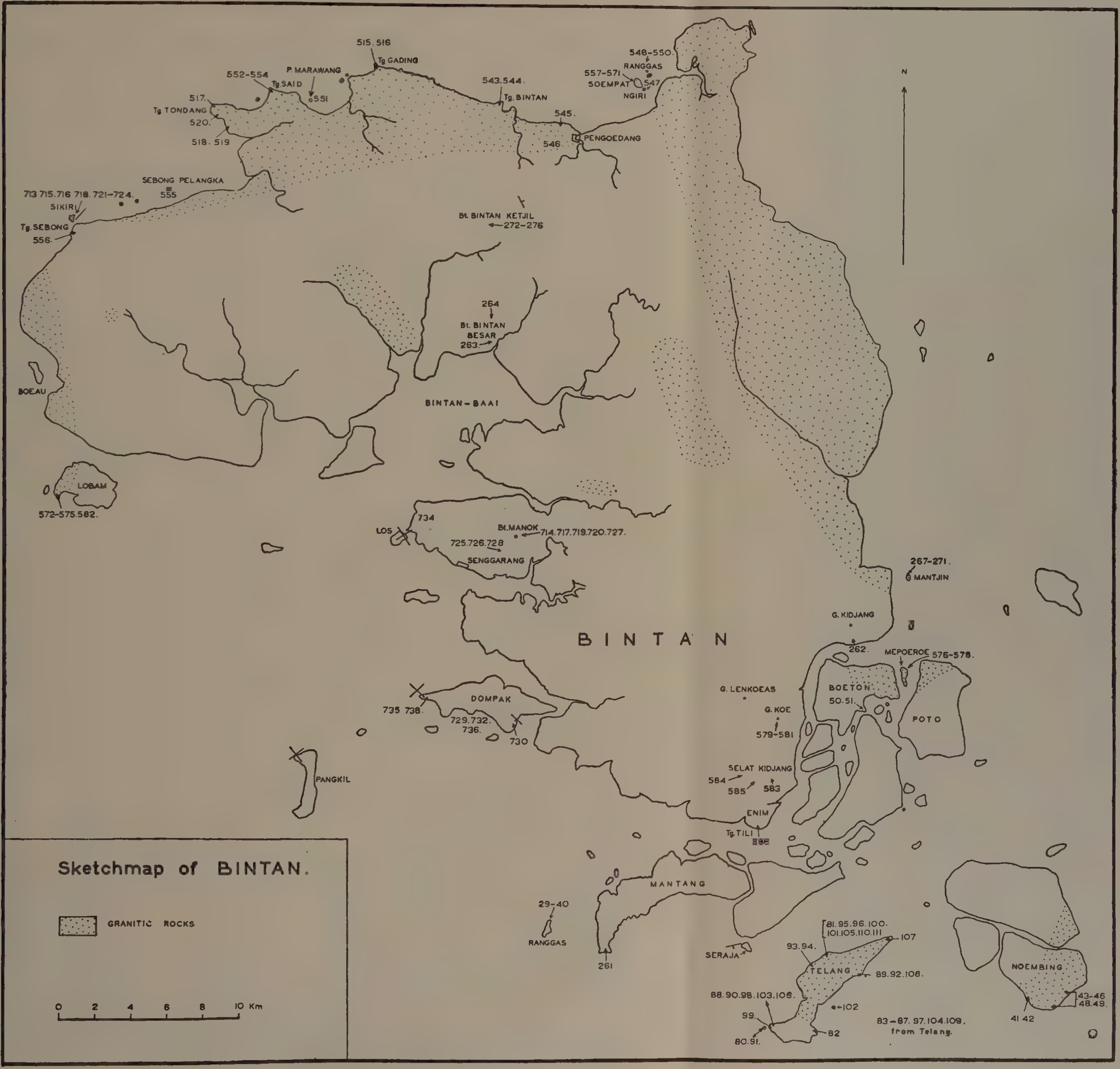


Fig. 4. 65 ×
Crossed Nicols.

Fig. 3, 4. Granophyric Rock, entirely hydrargyllitized.
Selat Kidjang, SE Bintang.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLV, 1942.

J. J. HERMES and D. R. DE VLETTER: CONTRIBUTION TO THE PETROGRAPHY OF BINTAN (RIOUW-LINGGA ARCHIPELAGO).



Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLV, 1942.

Geology. — *On rocks from Karimon (Riouw Archipelago).* By B. v. RAADSHOOVEN and J. SWART. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of December 27, 1941.)

Introduction.

The examined material has been given to the "Min.-Geol. Inst." at Utrecht by Dr. P. M. ROGGEVEEN, who in 1930, in charge of the Billiton Mij., made geological researches on some islands of the Riouw Archipelago. We are indebted to this company for having placed ROGGEVEEN's reports at our disposal.

For a list of publications concerning the tin-islands we refer to BOTHÉ (1). R. EVERWIJN assisted by J. A. FLEURY made during 1863/1864 explorations for tin-ore on some islands of the Riouw Archipelago (2). He established that the greater part of Karimon consists of granite and some protogene and a rock apparently a greisen. With the exception of quaternary deposits there are no sediments. As the rich tin-ore deposits of Banka are mostly found in sediments, he supposed that Karimon would prove to be poor in tin.

The more extensive report of A. CHR. D. BOTHÉ (1, 3) is based on observations by BOTHÉ, BOERS, DE KROES and LOTH. W. F. GISOLF determined the rocksamples. He concluded that the surface of Karimon has cut the upper part of a granite-batholith, consisting of granites, biotite-, aplitic biotite-, and tourmaline-granites and also of greisen. Sediments are found on the N.E. part of Karimon and on Karimon Anak. They are strongly contact-metamorphic slates, quartzites, cherts, conglomerates and limestones, enclosed as roofpendants in the granite. Possibly the contact-metamorphic limestones of Malarco are of the same age as the Raub Series of Malaya (Carboniferous).

According to ROGGEVEEN (in his Reports), the oldest formation is formed by the schistose hornblende-schists of P. Temblas and the amphibolitic rocks of P. Merak and Tandjong Malolo, which are probably synchronous with these schists. The rocks of P. Merak have been injected by granitic rocks (pegmatites, aplitic graphic granites etc.). He considers the rocks of Temblas as regionally metamorphosed sediments; the amphibolitic rocks of Merak and Tg. Malolo probably, according to him, belong to the same sediment-formation; they have been metamorphosed by the Karimon-granite.

The determination of the samples taught us, however, that all these rocks are not of para-, but of ortho-nature. Gabbros, altered gabbros and diallagites with their metamorphics could be distinguished. ROGGEVEEN's granitic injections proved to be gabbro-aplites and plagi-aplites.

As sediments, following in age on the amphibolites, ROGGEVEEN regards the strong contact-metamorphic calcareous and argillaceous sediments of Karimon Anak and Malarco. He places them in the Carboniferous, for their resemblance with the carboniferous limestones of Malaya. The only fossil found (Malarco) is a not determined coral. At Karimon Anak the sediments are changed contact-metamorphically by the granite into calc-silicatehornfelses. The contactzone is parallel to the strike of the sidements (N. 90 E., 13 N.). At Malarco these rocks occur as inclusions in a formation, called by ROGGEVEEN microgranite.

A sediment-formation that is probably triassic, on the base of fossile-finds in Malaya in a similar formation, is found on Karimon at the surroundings of Tg. Sebatak and at Malarco. The rocks of Tg. Sebatak are argillaceous and sandy; they are strongly weathered and show lateritization. Their strike is parallel to the coast. The contact with the granite is nowhere disclosed. The sediments of Malarco, the samples of which are lacking in our collection, are described as grey sandy shales with a general strike N. 160 E., dipping 27° to the W. They have not been changed by the microgranite, which occurs in the direct neighbourhood. ROGGEVEEN regards these sediments as older than the microgranite, because they occur in a topographic lower level than the surrounding rocks.

Post-triassic are the granites of these islands. The granite of Karimon and Karimon Anak consists of a biotite-granite of medium grain. At Karimon Anak a fine-grained porphyritic variety is also found. Basic differentiations have not been observed by ROGGEVEEN. The samples show us, however, that the rocks of Semamal and Semampang are altered quartz-diorites. We suppose, that they are probably a part of the marginal area of the granite-batholith. At different places aplitic complexes have been found (e.g. N. and S. of Semamal and N. Karimon). Aplite-veins are common in the granite (e.g. N. of Semamal and P. Moedoe). Sometimes the aplite-veins contain tourmaline, fluorite and sulphidic ore. Pegmatites and pneumatolytic rocks have also been found in the granites. At some places nests of quartz, tourmaline and muscovite occur (e.g. N. Karimon, Karimon Anak at the sediment-contact and S. of Semamal). Greisen is found at Upper Pelambong and S. of the Boekit Djantan.

The peninsula of Malarco and Karimon Anak partly consist of a microgranite with many inclusions of different origin. They are at Malarco porphyritic eruptives and contact-metamorphic ? carboniferous sediments, as mentioned above. Sometimes the inclusions show an arrangement parallel to the tectonic direction: the microgranite has possibly intruded according to the bedding, changing the sediments into exogeneous inclusions. BOTHÉ thought these microgranites to be the roof-breccia of the batholith; ROGGEVEEN supposed them to be a post-granitic intrusion.

Determination of these microgranites showed, however, that they are quartzporphyrites and fine-grained tuffs, up to agglomeratic tuffs of quartzporphyritic composition, with inclusions of metamorphic limestones at Malarco. Some of these rocks show pneumatolytic alteration: they contain many small tourmaline crystals. Therefore they may be regarded as older than the granite.

Description of rocks.

1. *The basic plutonic rocks with their veins and metamorphics, from P. Merak, P. Temblas and Tg. Malolo.*

   1a. *Amphibole-gabbros, amphibole-saussurite-gabbros and their veins* (215—220, 245—252, 455, 457, 462, 463, 495—503, 505—508, 615—617) 1).

This group contains fine-grained dark rocks and coarser grained darkgreen and white spotted rocks with large dull-grey parts and light-coloured veins. Microscopically there is, with exception of their crystal-size, no difference. They are holocrystalline, hypidiomorphic rocks, with as main constituents a basic plagioclase and amphibole, which alternate quite irregularly.

The plagioclase is always labradorite or labradorite-bytownite; the crystals are hypidiomorphic, broadly prismatic and sometimes allotriomorphic. There is often polysynthetic twinning; in many crystals the twinning-lamellae wedge out and crystals without twinning are also seen. The felspars are often bordered by a corona of very small light-green amphibole needles, which also occur as numerous inclusions in the felspar. Only exceptionally the felspar has been decomposed into sericite and kaolin; the most frequent alteration being saussuritization. Many felspars are clouded with zoisite-epidote grains, and some parts of the rocks — the macroscopically grey parts — contain only saussurite, with rare secondary quartz between the grains.

The amphibole is a green to lightgreen feebly pleochroitic variety. It appears in prismatic hypidiomorphic crystals, most of them badly terminated, and also in small needles and fibrous complexes. The larger crystals often contain inclusions of magnetite. A more

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1) The numbers of the samples correspond with those of the year-catalogue of 1941 of the "Min.-Geol. Inst." at Utrecht and with those of the map.

dark-green and strongly pleochroitic variety occurs in small veinlets in 220 and 616.

Apatite and zircon are accessories.

In 217, 220, 249, 250, 455, 498 and 507 we observe cataclastic phenomena. The felspars are crushed or the lamellation is bent; so is the amphibole.

It is evident that most of the gabbros are somewhat altered and it is sometimes difficult to state whether the rock is a gabbro or an amphibolite.

In 218 phillipsite occurs, crystallized in small fissures.

A vein of gabbro-aplite occurs in 508. It consists of andesine in hypidiomorphic crystals, some allotriomorphic quartz and some amphibole in needles and prisms; saussuritization is rather strong. The veins in 245 and 251 are plagi-aplites with quartz. The main constituent minerals are quartz in often strongly crushed grains and albite in better formed crystals, also often crushed or with bent lamellation. Amphibole needles as well as epidote and saussurite occur in small amounts.

The samples 223, 224, 227, 229, 234, 456, 509, and 510 are greyish-greenish rocks. They consist of irregular shaped quartz, in mozaic-like complexes and often strongly crushed and of felspar, which is mostly albite, with and without twinning and rather clear; in few cases there occurs oligoclase. Zoisite is very abundant in large crystals and as a component of granular complexes; amphibole is rare. In 227 and 509 there occur large rock inclusions, consisting wholly of amphibole crystals; they are regarded as altered inclusions of the bordering gabbro. All these quartz-albite-zoisite rocks probably are altered plagi-aplites, or, when there occurs much zoisite, strongly decomposed gabbro-aplites.

221, 222, 225, 226, 230, 231 and 238 are amphibolitic rocks. The configuration of the amphibole and felspar (a labradorite) is not gabbro-like. The minerals group together and the amphibole crystals are much larger than in the gabbros. The felspar has for the greater part changed into saussurite; secondary albite occurs between the zoisite grains. Because of their mineral composition and their occurrence together with gabbros and changed gabbros, it is almost certain that these rocks are ex-gabbros.

1*b*. *Uralite-diallagites* (228, 232, 233, 241, 504, 505).

These rocks are dark-green and wholly built up by large crystals. Microscopically they consist of large diallage crystals, prismatic, with distinct rectangular cleavage and feeble pleochroism. They are strongly uralitized, whole parts are changed into green amphibole and sometimes into actinolite. The uralitization starts at the terminal ends of the crystals and there often remains a core of unchanged diallage. Inclusions of many small grains of titaniferous ore are situated in the directions of cleavage. There occur also small amounts of sulphidic ore.

1*c*. *Amphibole- and actinolite schists* (235—237, 239—244).

The only constituent of these rocks is a green, pleochroitic amphibole, in large tabular crystals, with inclusions of magnetite and in fibrous complexes. Instead of amphibole feebly coloured actinolite may occur, often in fine, nephritic aggregates (239, 241, 244). Between the amphibole or actinolite there is a small amount of zoisite grains; in 237 zoisite occurs in a veinlet.

It is very probable that these rocks are metamorphosed diallagites.

1*d*. *Schistose amphibolites* (606—613).

These dark-green rocks are schistose and some are finely foliated (608, 610). Microscopically the schistosity is clearly pronounced by the parallel orientation of the minerals.

Between long prisms and needles of a dark-green pleochroitic amphibole, there occur small zones with orientated clear albite crystals, without twinning, and some quartz grains. If the amphibole occurs in larger crystals, they are often broken. Zoisite and epidote grains are also distributed in parallel zones (608, 609, 613). In 610 there occur saussurite and kaolin. In all rocks there are small grains of sphene and ore. In 607, 612, and 613 veins are cutting through the schistosity, containing albite, quartz, chlorite and zoisite.

It is very probable, that these rocks are ex-gabbros, changed by regional metamorphosis in the meso-zone.

2. *Contact-metamorphic ? carboniferous sediments from the granite-contact.*

2a. *Calc-silicate-hornfelses, all from Karimon Anak* (363—372, 377, 379, 398).

These rocks are well stratified. Different layers are to be distinguished: *a*) green bands with columnar crystals and round grains, *b*) blue-grey compact layers, which contain a sulphidic ore, *c*) light-coloured, pink, hard bands, consisting of pink grains with a lustre of glass, situated in a light-coloured groundmass. Microscopically the bounderies of the bands are not sharp.

The pink layers of the rocks mainly consist of garnet (grossularite), mostly in not well-bounded complexes. Among the garnet-groups and as inclusions wollastonite occurs; the crystals are small and twinned. Prehnite is found just like wollastonite. A little calcite and some grains of quartz are lying among the garnets. Besides garnet, several rock-samples contain vesuvianite, sometimes typical zonal. The garnet-complexes contain grains with high-refraction, undeterminable. Some samples show some Mg-diopside.

The components of the green-coloured layers are variable. Most of the samples (371, 363, 365, 369, 377, 379) contain vesuvianite, showing anomalous interference colours and sometimes beautiful zoning (365). On the other hand 378 contains some xenomorphic quartz crystals, besides prehnite. The sample 368 shows a groundmass of wollastonite, which is often twinned, enclosing rounded grains of garnet. Other rock-samples contain garnet, wollastonite, prehnite, calcite and quartz in variable quantities.

The compact blue-grey layers (364, 372) mainly consist of wollastonite and prehnite, and small quantities of quartz, garnet, calcite and ore grains. The sample 370 mainly contains calcite with mosaic- and partly interlocked structures. Moreover garnet, vesuvianite, quartz and ore occur.

2b. *Hornfelses with prehnite, wollastonite and garnet, from Karimon Anak* (373, 374).

These are dark-grey, finely-stratified rocks, consisting for the greater part of a finely-crystalline groundmass, composed of quartz and an acid, lath-shaped felspar. Some layers contain prehnite and wollastonite; others are composed of garnet with inclusions, prehnite, wollastonite and undeterminable grains with high-refraction. Further sericite, sulphidic ore and epidote occur.

2c. *Mica-hornfelses, all from Karimon Anak* (375, 376, 378).

Compact stratified rocks, consisting of small quartz-grains, acid lathshaped felspar and sericite. In small quantities muscovite, chlorite, sulphidic ore and limonite have been found. Accessories: apatite-needles.

3. *? Triassic rocks from Tg. Sebatak, S. from Kasiabang, P. Assan and P. Temblas* (451, 494, 603, 604, 614).

In his Reports ROGGEVEEN mentions the occurrence of Trias in a small outcrop from S. Malarco and from S.E. Karimon (Tg. Sebatak). From the first locality there are no samples in his collection. From the second locality there are two samples (451 and 604). 604 is a strongly limonitized and hydrargillitized rock, which contains large crystals of quartz, giving the impression of phenocrysts, and hydargillite-complexes, which may be pseudomorphic after felspar. The rock is probably an altered quartzporphyritic tuff. 451 is a strongly limonitized rock with many splinters and grains of quartz. It is possible that the quartzs are porphyritic ones, but this is much less certain than in 604. We may call the rock a limonitequartz sandstone.

The character of these rocks does not at all prove that they are of triassic age. It may be that they belong to the tuffs of E. Karimon, which will be described afterwards.

In the collection there are two rounded pebbles from P. Temblas (494, 614), which are very similar to the graywacke sandstones of propably triassic age from Soegi [1]). They contain as clastic grains: quartz (often cataclastic), quartzite, ? chert and sericitized fragments; moreover rare felspars. They contain veins with sericite; the rocks therefore must have undergone sericitization.

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[1]) These Proceedings, XLIV, 1941, p. 1223.

One sample, seemingly from an outcrop at P. Assan (603) is a limonitic quartz-sandstone; the clastic components are quartz and muscovite. On the map only granite has been indicated at this locality.

4. *Quartzporphyrites and quartzporphyrite-tuffs, more or less contact-metamorphic, from Karimon Anak, Malarco and Tg. Batoe Besar* (361, 362, 385—388, 391—393, 396, 397, 399—415, 417, 431, 443—445).

These rocks are both macroscopically and microscopically very variable. The general characteristics are often dimmed by secondary processes, which almost certainly are due to contact-metamorphosis.

4a. *Quartz-porphyrites* (392, 393, 400, 408, 415, 425, 426, 430, 431).

These rocks consist of a groundmass, containing very fine grained quartz and very small, undeterminable felspars and distinct phenocrysts of quartz and plagioclase. The quartz-phenocrysts are clear, hypidiomorphic and often strongly corroded. The felspar, in more or less idiomorphic crystals, often with lamellar twinning, has a composition ranging from albite-oligoclase to oligoclase. Silicification of the groundmass and sometimes also of the felspar-phenocrysts occurs in the samples 400, 408, 415, 425, 426, 430, and 431. In 426 there are no quartz-phenocrysts, whereas the rock is prehnitized and epidotized. 392 and 393 contain also prehnite and epidote; they are in contact with a "quartz-prehnite-hornfels": a dark, fine-grained rock, containing quartz-grains, rather many prehnite-strings and some calcite. These hornfelses are probably metamorphic, fine grained tuffs. 408 is pneumatolytically altered: numerous small tourmaline-crystals and sometimes a tourmaline-sun occur in the groundmass. Epidote, zoisite, calcite and apatite are accessories.

4b. *Quartzporphyrite-tuffs.*

These tuffs show microscopically rather strong variation with regard to structure and composition. They range from a very fine recrystallized ash-tuff (414) to crystal-tuffs (401, 402, 404, 406) and pass gradually, with the strong increase of rock-fragments and decrease of "groundmass" into coarse agglomeratic tuffs (361, 362, 443, 444, 445).

The "groundmass" consists always of fine grained quartz and felspar. At the side of this "groundmass" occur many larger fragments of quartz and plagioclase. The quartz is clear and the form of the fragments is angular: they are splinters of phenocrysts. The felspar is hypidiomorphic and most times strongly sericitized and epidotized; the composition varies from albite to albite-oligoclase. The rock-fragments are almost always of quartzporphyritic composition: a groundmass containing plagioclase-laths and quartz, with quartz and albite-oligoclase phenocrysts. Sometimes there occur fragments of a strongly silicified groundmass or wholly epidotized fragments. In 443 a xenolithic quartzite fragment (!) has been found. Silicification of the "groundmass", the felspar-fragments and the rock-fragments is often observed. This is prominent in the samples 385, 386, 397, 399, 405, 406, 413, 426, 428, 430. The felspar of the "groundmass" has disappeared, whereas at some places the "groundmass" becomes quartzitic; the felspar-fragments and the plagioclase-phenocrysts of the rock-fragments change into fine-grained quartz, but preserve their primary form. Sometimes there is a rather strong prehnitization of the rocks. This is very evident in 387, 388, and 426. Secondary prehnite crystals are scattered throughout the whole "groundmass"; they too occur grouped together and form rather large complexes; they are accompanied by epidote-zoisite, some diopside, scarce wollastonite and calcite. Actinolitization is seen in 444, 445 and 361. The actinolite has crystallized in small needles and sometimes there occur nests, containing larger crystals. 401, 404, 406 and 444 have been pneumatolytically changed. Throughout the whole "groundmass" many small tourmaline-crystals and needles are found. 428 is biotitized, whereas 427 and 428 contain small chlorite crystals. In almost all the samples there is some secondary epidote-zoisite. Magnetite, sulphidic ore, apatite, zircon and some sphene-grains are accessories.

A typical group is formed by the tuffs of E. Malarco, Karimon (407, 409—413, 417—421 and 423). With the naked eye we can already distinguish large inclusions of crystalline limestone, bordered by a green mineral. The limestone has been changed into epidote-marble: large irregular calcite crystals, between which occur many epidote grains, diopside, prehnite, wollastonite, quartz and a dark-green strongly pleochroitic amphibole. 420 is an inclusion, wholly formed of large prismatic wollastonite crystals with anomalous garnet (grossularite) and other contact-minerals. All these limestone-inclusions are bordered by a reaction-zone, containing dark-green, pleochroitic amphibole, in prisms and needles. Prehnite occurs in large allotriomorphic crystals. At some distance of the contact with the limestone, we see the ordinary tuff-configuration, but there remains always a little calcite, amphibole and prehnite, scattered through the slide.

5. *Granitic rocks from Karimon, Karimon Anak, P. Moedoe, P. Assan and P. Tengkorak.*

5a. *Granites* (180, 183, 186, 380—382, 384, 389, 390, 394, 395, 424, 442, 448, 450, 452, 454, 458, 465, 466, 472, 475, 479, 481—487, 493, 587—590, 595—600, 602, 605, 657—659, 662, 663).

The samples of these light-coloured rocks show a varying grain-diameter. The texture of 658 is porphyritic; 383, 466 and 590 show pegmatitic intercalations. Clear quartz, white felspar and dark mica are macroscopically visible. Microscopically the greater part of the rocks prove to be biotite-granites and their aplitic varieties. Beside them, bi-mica granites (454, 482) and a number of aplitic granites occur. The rocks mainly consist of quartz, orthoclase, biotite and in some samples a small quantity of muscovite. Quartz shows xenomorphic development, with undulatory extinction and cataclastic zones. In a number of rock-samples graphic intergrowths with orthoclase have been found (381, 383, 390, 395, 398, 483, 493, 587—589, 596, 597, 663). Orthoclase generally forms large crystals, often with perthitic textures. Sericitization occurs in many cases. Plagioclase (albite up to albite-oligoclase) appears as small crystals with twinning lamellae which are sometimes bent. Biotite, in green, frequently bent crystals, is often bleached and transformed into chlorite. Accessories: apatite, zoisite, zircon and leucoxene. A number of rock-samples are stressed (180, 442, 454, 479, 483, 486, 487). The features are: cataclastic quartz and bent crystals of plagioclase and dark mica. A part of the samples contain pneumatolytic minerals: fluorite, in grains and complexes (450, 454, 479, 481, 483, 485, 587, 595, 598—600, 602, 659), topaz (390, 479, 481, 483, 598, 658), tourmaline (481, 484, 485, 596, 598, 599, 605), cassiterite (481, 596). At the contact between the biotite-granite and hornfels (380—382, 395, 663), the crystals become smaller and are orientated perpendiculary to the contact.

5b. *Granite aplites* (183, 184, 188, 375, 435, 438, 441, 449, 453, 457, 461, 464, 473, 474, 476, 480, 488, 591—594, 601, 660, 661).

Light-coloured, fine-grained rocks, sometimes with pegmatitic intercalations (474, 476) and with a porphyritic texture (461). The composition of these rocks is quite the same as that of the granite, only the quantity of the biotite is smaller. Most of the samples are a little stressed (quartz with undulatory extinction). A number of these rocks contain pneumatolytic minerals: fluorite (441, 473, 474, 476, 480, 488), topaz (188), tourmaline (473, 599, 661). One of the samples (488) shows a transition between a granite aplite and a mica-quartz rock (both with fluorite).

5c. *Granite pegmatites* (191, 465, 475, 489, 662).

They contain the same minerals as the granite, with the exception of biotite. In a few samples muscovite is found, others show a strong kaolinization (191, 489). Some rocks contain pneumatolytic minerals: tourmaline (191), fluorite (475).

5d. Greisen (185, 187, 189, 190, 192, 193, 432, 434, 437, 446, 447, 477, 492, 511).

They are to be distinguished after the composition into:

a) *Mica-greisen* (192, 434, 477, 492). Light-coloured, coarse-grained rocks, consisting of clear quartz-grains lying in a white powder-like mass, besides of muscovite. Quartz occurs in large xenomorphic crystals, mostly with an undulatory extinction and also in little grains. Further occur muscovite in little scales and biotite (434). Fluorite is found in some quartz-grains (477, 492). Accessories: magnetite, zircon and topaz (434);

b) *Topaz-greisen* (185, 187, 437, 511). Light-yellow, coarse-crystalline rocks. Quartz and black ore are macroscopically visible. Large xenomorphic quartz crystals often show undulatory extinction. Further, accumulations of small topaz-grains and colourless mica-scales as inclusions in quartz, and topaz occur. The crystals of the mica are often bent. Ore and limonite are present in different quantities. Accessories: tourmaline, biotite and cassiterite.

c) *Tourmaline-greisen* (189, 190, 193, 432, 446, 447). Hexagonal tourmaline-columns, pleochroitic and sometimes with zonal arrangement, occur in varying quantities. Moreover the rocks contain quartz, a small amount of colourless mica, fluorite (189, 190), topaz (432, 447), cassiterite (189, 432) and felspar-fragments (447).

5e. Tourmaline-mica-rock (478).

The rock-sample is mainly composed of dark columnar tourmaline and much scaly and radiated mica. Further we find green chlorite and limonite.

5f. Mica-rock with tourmaline (440).

This sample mainly consists of mica. Besides, tourmaline-columns and some quartz-grains occur.

5g. Quartz diorites (467, 468, 470, 471).

Strongly-weathered, grey-green, coarse-grained rocks. They consist of allotriomorphic to hypidiomorphic plagioclase (albite up to albite-oligoclase). Twinning lamellae are badly developed and the crystals are usually broken. The felspar is often chloritized and sericitized. The second main constituent is chlorite, in large, compact, green, fine-grained complexes. A little quartz with undulatory extinction has been found. All the samples contain fluorite in different quantities and 468 biotite. Sericite occurs in strings; further a small amount of zircon, magnetite and limonite appears.

5h. Quartz with fluorite (490).

Consists of large, interlocked quartz-grains and aggregates of small grains. The second constituent is formed by fluorite, the crystals show a good cleavage. Further in small quantities limonite and muscovite.

5i. Brecciated quartz-rock (433).

The rock mainly consists of large, mostly broken and small angular quartz crystals. At some places zonal structures have been found. Also a little limonite and sericite occur.

SUMMARY.

The metamorphic basic igneous rocks of Tg. Malolo, Kasiabang, P. Merak and P. Temblas probably belong to the oldest formation. Their schistosity at P. Temblas and their minerals point to metamorphic alteration in the meso-zone.

The strongly contact-metamorphically altered limestones on Karimon Anak resemble the carboniferous rocks of the Raub Series on Malaya. To the same formation probably belong the inclusions in the effusives at Malarco. Similar inclusions may have existed in the tuffs of Karimon Anak; they may have been resorbed by following contact-metamorphosis by the granite. This process would explain the abundance of prehnite in these tuffs.

Most probably the quartzporphyrites and their tuffs on Karimon Anak and Malarco, called by ROGGEVEEN micrograpites, are of prae-granitic age: they have been contact-metamorphically and pneumatolytically altered by the granite. The effusives contain at Malarco the above mentioned inclusions of ? carboniferous rocks. These porphyrites and

tuffs resemble components of the Pahang Volcanic Series; accordingly, their age may be carboniferous to triassic.

We feel not at all sure that the rocks regarded by ROGGEVEEN as triassic, indeed belong to that formation. The only rocks of any importance we could study, were the limonitized sandstones from Tg. Sebatak, which possibly are altered tuffs.

The granite of the islands in the Riouw Archipelago is regarded to be of post-triassic age; although nowhere a contact with the triassic sediments is exposed, the granite of Karimon may be of the same age. The batholith is composed of feebly-stressed biotite-granite, with aplitic- and pegmatitic varieties. All over the island pneumatolytic alterations of the granite (formation of greisen and the occurrence of tourmaline, topaz, fluorite and cassiterite) are observed.

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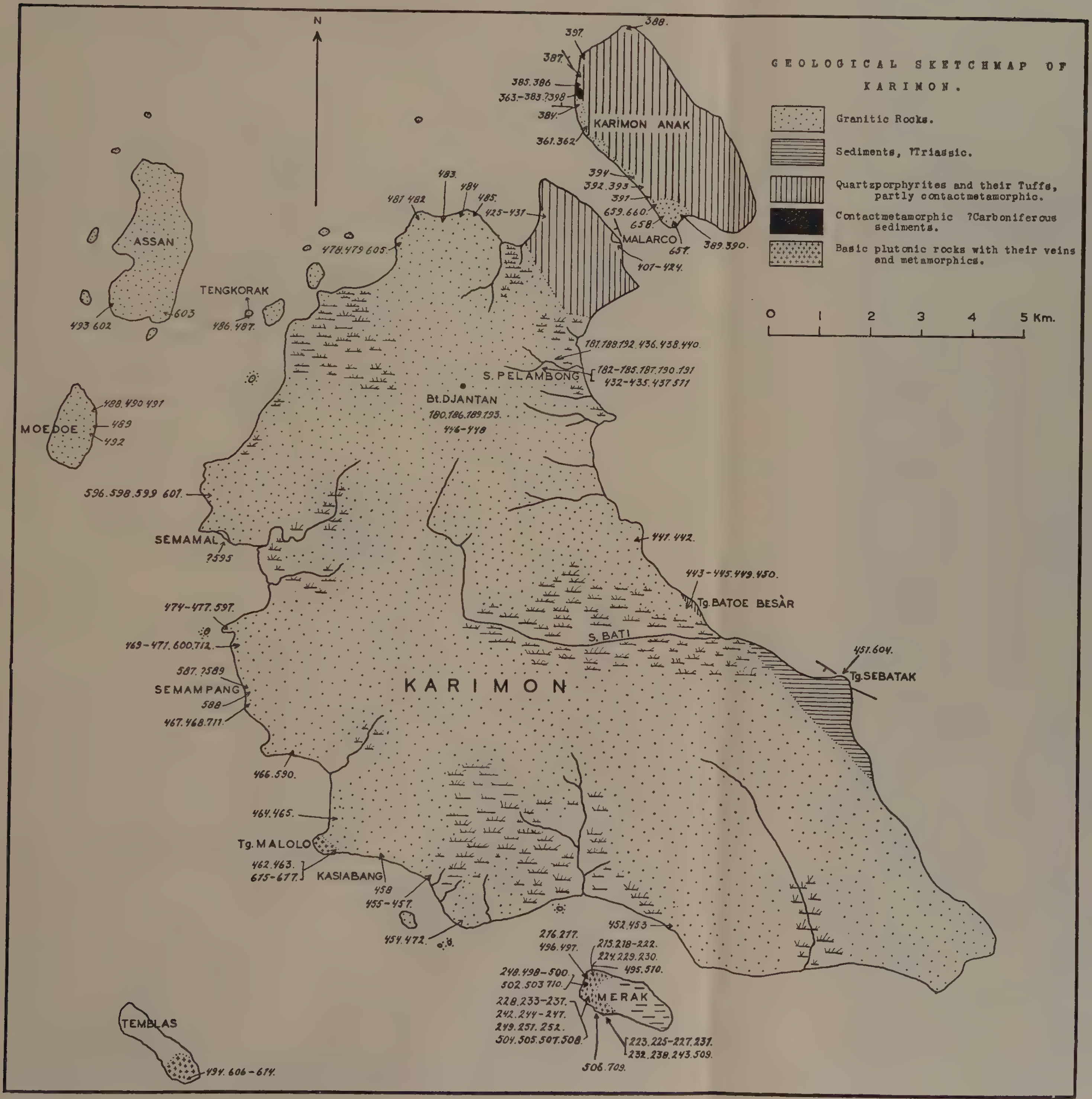
B. v. RAADSHOOVEN and J. SWART: ON ROCKS FROM KARIMON (RIOUW ARCHIPELAGO).



Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLV, 1942.

Zoology. — *New observations on the feeding of Vampyrella lateritia (Fres.) Leidy.*

By H. R. Hoogenraad and A. A. de Groot. (Communicated by Prof. J. Boeke.)

(Communicated at the meeting of December 27, 1941.)

(*Rhizopoda* and *Heliozoa* from the freshwater of the Netherlands. IX).

In sphagnum which we gathered on the "Zijpenberg" near Rheden in April 1941 we found in May and June of that year a number of specimens of a *Vampyrella*-species which was most probably *Vampyrella lateritia* (Fres.) Leidy.

They fed on the contents of the cells of a *Mougeotia*- (= *Mesocarpus*- ?) species. A few times we were able to watch this process closely; as it differed in some points from what was found by the former observers — Cienkowsky, Penard, West, Cash, Lloyd, Gobi — and we could moreover compare it with our own observations both on *Vampyrella* and on *Hyalodiscus* and *Difflugia*, we thought it worth while to give a more or less detailed account of our experiences, the more so as the descriptions of the observers mentioned above do not agree with each other in every respect and this species is relatively seldom observed.

The morphology of the animal showed the normal characteristics (Fig. 1). During the rapidly and evenly gliding motion between two feedings the form was of a moderately flattened spheroid, seen on top a pure circle or a broadened ellipse; this last form was more prominent in specimens that crept forward along the weed-filaments. The size without the pseudopods was from 20—50 μ. Ecto- and endoplasm were distinctly separated; the first formed a narrow, colourless border round the second, which as a rule showed the normal brick-red colour, sometimes with clearly visible green spots inside, caused by the food taken. A few brownish-green specimens were seen, a colour probably caused by the ingesting of the contents of cells with their chromatophores repeatedly and at short intervalls. Sometimes a bright central spot was visible in the thick endoplasm, not sharply separated from the surrounding plasma, probably the place where the nucleus was found. Contractile vacuoles could not be established with certainty, though a number of non-contractile ones could be seen, bulging outside the ectoplasm-border as small bubbles.

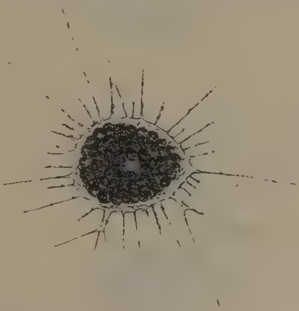


Fig. 1.

The pseudopods, mostly radiating on all sides, appeared in the two forms usually distinguished in this species: the long ones, sometimes three times the diameter of the plasm-body, either branched or not, with distinct grain-movement, and short ones, so-called pinhead-formed, distinctly thickened at the top, disappearing and appearing again and again. The nature of the last is still problematic; in some cases we got the impression that the sudden darting of this pseudopods was not what it seemed to be, but found its cause in reality in the rapid gliding of a granulum along an already existing pseudopodium of the first kind, of which the distal part had escaped our observation. In a single case we saw a pseudopodium, more than 160 μ long, which with a broad basis as a sort of pseudopodium-stalk rose from the plasm-body and ended in two fine threads, which the animal cast as an anchor on the objectglass while it ingested its food (Fig. 1c).

Now at the hand of the figures we shall first give an account of our observations on

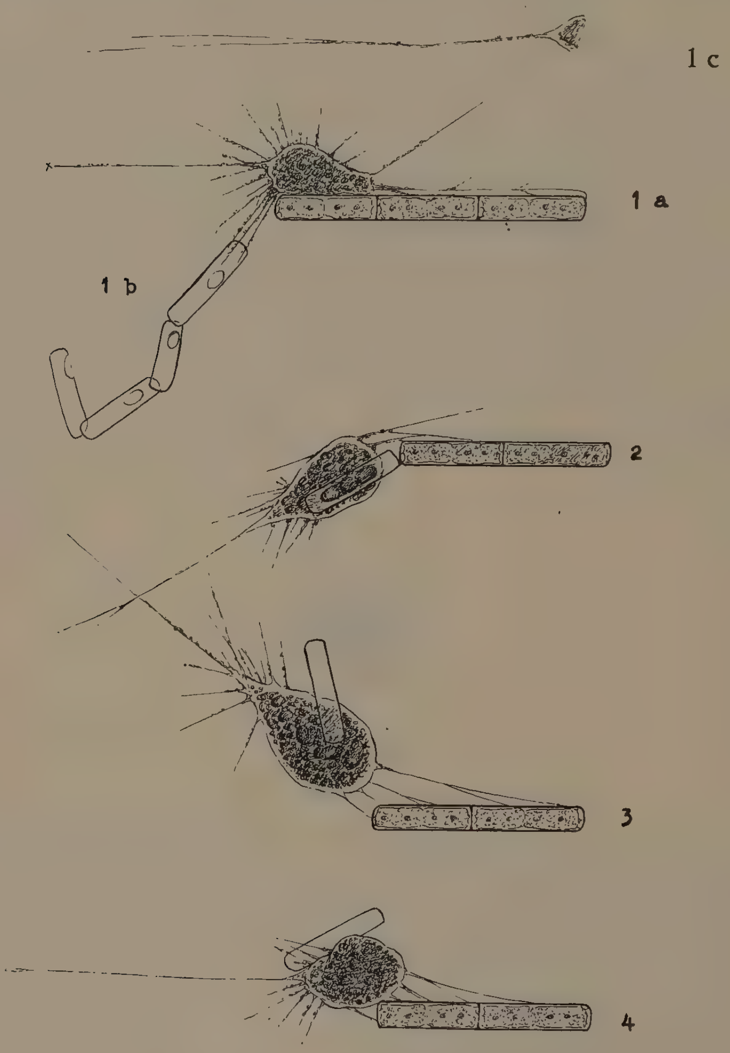


Fig. 1a—4.

the feeding and then shortly relate the experiences of earlier observers and our own (1907 a and b). We repeatedly observed this feeding, which on the whole followed the same lines, and here give the descriptions of two cases which may be considered as typical.

June 1. 9.10 a.m. (Fig. 1a, b). A *Vampyrella*-specimen is seen in contact both with a filament of a *Mougeotia*-species, consisting of three short, living cells, and with a series of four empty cells, connected together, which all showed the typical round hole, through which they had been emptied by a *Vampyrella*-specimen, probably by the same that was on its way to the three-cell-filament, with which it was already in contact. The plasm-body, somewhat long-stretched now, still possesses pseudopods that radiate on all sides; at the side that led the way in moving they were united to a thick bundle, which also stretched itself on the two following cells, to which these pseudopods attach themselves like "pulling-threads". At the back of the animal is visible the above-mentioned pseudopodium, 160 μ long; this remains there during the whole process of ingestion of the food.

9.15 (Fig. 2). With a sudden shock the cell thus attacked breaks loose from the others, while the animal swings with this cell round the above-mentioned pseudopodium, which acts as an "anchor-cable". The pulling-threads remain in contact with the two other cells; simultaneously with the shock the now isolated cell begins to empty itself rapidly.

9.17 (Fig. 3, 4). The first cell is entirely empty now; the empty cell-wall sticks for some time to the animal. The pulling-threads to cell 2 and 3 are shortening.

9.20 (Fig. 4a). While it continues to shorten the pulling-threads the animal lies down against the following cell; one of these threads clearly embraces the top of cell 3.

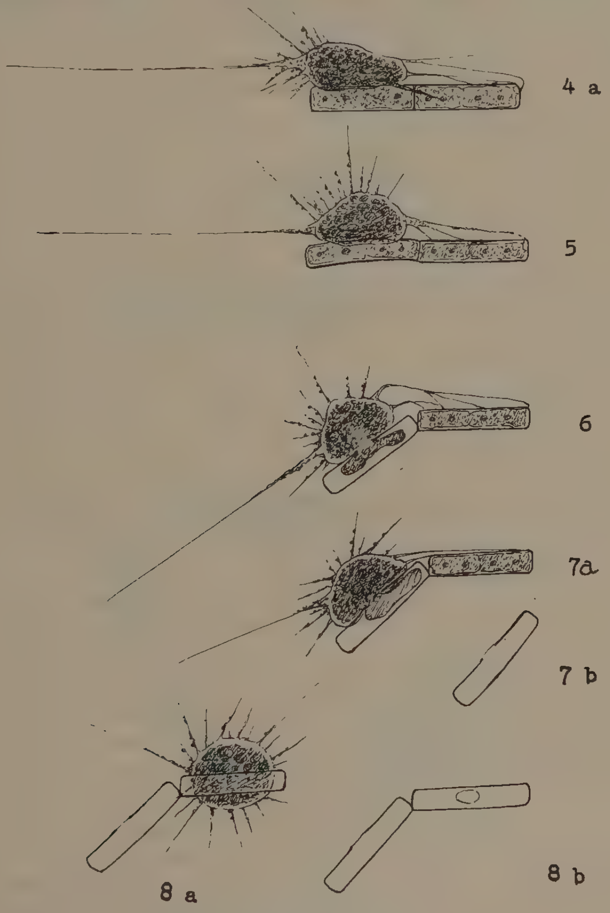


Fig. 4a—8b.

9.25 (Fig. 5). Cell 2 now seized shows a slight bending towards the side of the animal; then this cell too immediately bursts open and its contents disappear into the plasm-body of the animal. This is again accompanied by a shock, which tears the cell loose from the next, number 3, and makes the animal together with the cell attacked swing round the "anchor-cable", that is to say: flings it aside. Both cells (Fig. 6) remain connected with a small part of the cell-walls facing each other, just as was the case with the four empty cells which we saw first. Curved pulling-threads form a bridge between the two cells.

9.27 (Fig. 7a, b). Through the hole made in the cell-wall the animal puts a broad, finger-formed pseudopodium into the lumen of the cell as if to take up the remains of the contents possibly left behind ("to lick the pan clean"). After this the animal passes on to cell 3, which is emptied in the same way in about five minutes. The connection with cell 2 is maintained; the pulling-threads are no longer to be seen now, perhaps because we could only observe the animal from the bottom-side and not sideways as was the case with the two preceding cells.

9.35 (Fig. 8a, b). The last cell is left; the empty cells show the holes in the wall. The holes have an elliptical form and smooth edges without any trace of irregularities or fringes.

June 2. 5.30 p.m. The same specimen of the preceding day is observed while it is emptying a cell of a *Mougeotia*-filament, consisting of five long cells. The two cells following the attacked one are both still intact; after these two more cells follow, which show the typical holes in the wall and have probably already emptied been by the same or by some other specimen.

5.33 (Fig. 9). The animal leaves the final cell now emptied and proceeds to the

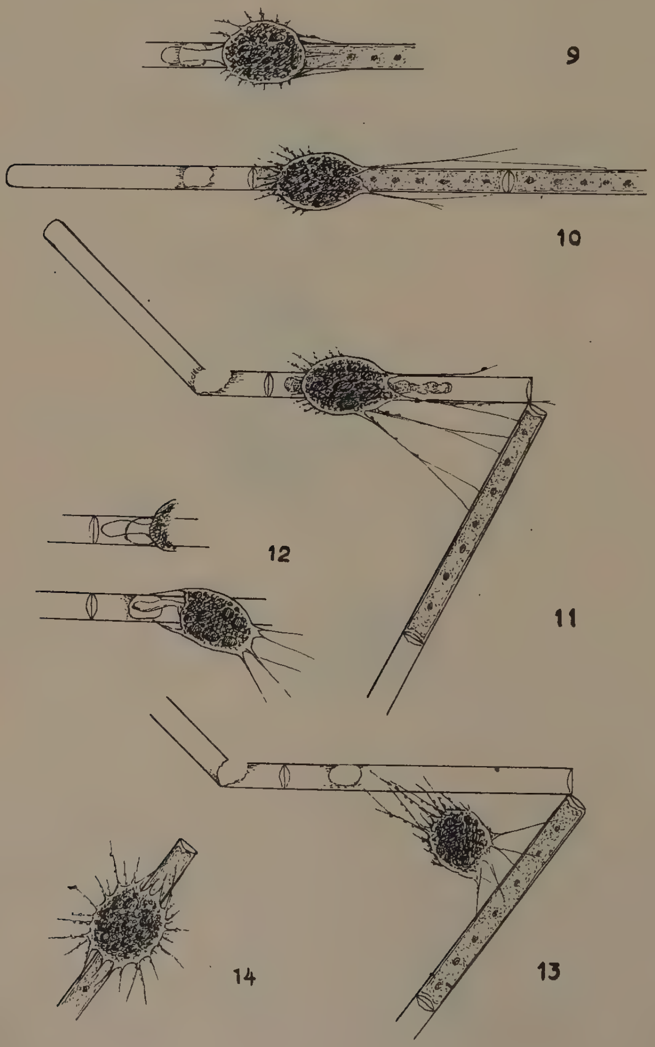


Fig. 9—14.

following. From the hole in the wall of the first cell, a broad, finger-formed pseudopodium is gradually drawn back, which was not visible before the animal moved away from the hole.

5.35 (Fig. 10). The contact with the first cell is totally lost now and the pseudopodium drawn back. Long pulling-threads stretch over the whole length of the second cell and reach as far as the next.

5.38 (Fig. 11). A violent shock shakes the whole weed-filament, the following things occurring simultaneously: the first cell shuts itself in the place where the hole is; the second separates itself from the third cell, which is still intact, at which the filament between cell 2 and 3 bends through, so that the two parts form an angle of ± 75°, and lastly the contents of the cell are made to a ball and devoured. The whole complex of processes has a violently explosive character.

5.40 (Fig. 12.) After the cell has been emptied a finger-formed pseudopodium appears, which behaves in exactly the same way as in the other cases.

5.47 (Fig. 13). The animal changes cell 2 for cell 3 and hangs between the two cells on his two bundles of pulling-threads.

5.48 (Fig. 14). The animal has attached itself to cell 3; the pulling-threads, very much shortened, are only fixed to this cell; there is no longer any contact with cell 2, just emptied.

5.50. The contents of cell 3 are suddenly poured into the plasm-body of the animal, where they become visible in the middle of the brown-red plasm as a green pellet. The situation between the cells of the filament remains unchanged.

We shall now give a short summary of what former observers relate about the feeding of *Vampyrella lateritia*. CIENKOWSKY (1865) is the first to describe the process, shortly, it is true, but accurately. His information comes to the following. After a *Vampyrella*-specimen has attached itself to a weed-filament — in this case of *Spirogyra* spec. — a few minutes' interval follows. Then suddenly the *Spirogyra*-cell thus attacked is seen to shift its place with a shock and at the same time to let loose the contents from the cell-wall. Shortly after this the contents pass very slowly into the plasm-body of the *Vampyrella*. After this the animal glides to the next cell, which is emptied in the same way. Thus cell after cell is robbed of its contents, till the *Vampyrella* entirely filled with food attaches itself to a weed-filament and encysts itself to digest the absorbed food. The ingestion of the cell-contents lasts 12 minutes on an average; during this time the pseudopods are either stretched out or they disappear entirely. The holes on the cell-wall are large but not clear-cut. As the animal has no hard parts we must take for granted that it is able to dissolve the cellulose-walls. Without any doubt it makes in this a deliberate choice: never did it attack a *Vaucheria* or an *Oedogonium*, even when these were purposely put before the animal; neither was food taken by total envelopment of a weed-cell.

PENARD (1889) remarks that his observations diverge from the usual opinion about the food-ingesting. This opinion means that the *Vampyrella* bores a hole into the wall of a *Spirogyra*-cell and introduces into it a pseudopodium whose task it is to seek the contents of the cell. Then PENARD himself gives the following description of the process, which he has repeatedly observed and always with the same results. After the animal has attached itself to a cell of the *Spirogyra*-filament and has drawn its pseudopodia an interval follows, during which nothing happens (so it seems at least). Then the central part of the plasm of the animal that is connected with the cell-wall withdraws from it while the outer part, in the form of a ring, remains attached to the wall; thus a vault-like cavity ("voûte") is formed in the plasm-body which rises higher and higher, till suddenly the cell-wall bursts ("se crêve") and the whole contents pass into the plasm-body. Now the animal stretches its pseudopodia again and goes away, while in the cell-wall a clearly visible tear ("déchirure") is left behind. The same author (1922) calls it "un petit trou non pas arrondi comme l'exigérait la théorie d'une dissolution de la cellulose, mais inégal et le plus souvent en étoile". (This, however, seems to be the outcome of later investigations, not published till yet). A second, sometimes also a third cell is emptied in the same way, after which the animal encysts itself. At last PENARD says once more emphatically that the ingesting of the food by *Vampyrella* takes place by means of a genuine sucking-process, in which the whole body of the animal functions as a suctorial organ ("ventouse"). A difficult coincidence is the fact that the walls of the weed-cell, as it is being emptied, do not fall together through the pression of the surrounding liquid, but, says PENARD, it is possible that, while its contents are disappearing, the cell fills itself with water through the wall.

The specimens of *Vampyrella* studied by WEST (1901) fed on the contents of *Mougeotia*-cells. He describes the process in the following way. The animal attached itself to a weed-filament and the long pseudopodia are drawn in, while at the same time from the ectoplasm shorter and broader pseudopods appear and are drawn again. Very soon after this the cell-wall is bored through and a part of the plasm-body of the animal enters the cell, causing a violently dancing movement of the granula of the vegetable cell. Now the *Mougeotia*-chromatophore begins to disintegrate at a point situated opposite the place of contact. In two hours only a part of the chromatophore and the surrounding plasm is absorbed by the animal; the final stadia of the process are not related. WEST further remarks that some observers have stated that *Vampyrella* does not perforate the cell-walls of the weeds, "but attacks them and devours their contents by breaking the filaments at the joints. It is possible that it does so sometimes, but CIENKOWSKY's original observation of the perforation of the cell of *Spirogyra* by this animal is, however, amply confirmed by the cells of *Mougeotia*, a plant which breaks at the junction of the cells much more readily than *Spirogyra*".

CASH (1905) says that his observations differ from CIENKOWSKY's, but agree with those of the later authors. After his statements the animal fastens itself with its longer and more movable pseudopodia to one of the cells of the weed-filament, *Spirogyra* or a related form, usually to the last cell. These long pseudopods have a remarkable force of contracting: they gather into a bundle on the side of the body where their presence is most urgently required. Through an effort scarcely intelligible in such a small creature, the filament on one of the joints is broken off and thus the animal gains admittance to the interior of a cell of which the contents are rapidly devoured by means of one or two finger-formed pseudopods that penetrate into the cell. As a peculiar case it is stated that a *Vampyrella*-specimen after emptying the last cell of a filament isolates the following cells one by one, through which proceeding they lie aside loose from one another, forming more or less straight angles which each other; every second of these cells is emptied then.

LLOYD (1929) first gives a summary of the observations of the Russian investigator GOBI (1925) [1]. GOBI's opinion about the process is as follows. After the animal has fastened itself, it begins to "show strain", which has the following consequences: 1. a big vacuole rises in the plasm-body, destined to receive the food, the "food-receptive" vacuole (= the "voûte" of PENARD ?); 2. the animal often jerks the filament and breaks it to pieces, or tears a single cell loose from the contact with the others. The last thing is often the case with the genus *Mesocarpus,* but not with *Spirogyra,* which proves that the cohesive force of the cells of the filament in the latter genus is greater than in the former. The food-receptive vacuole causes plasmolyse in the weed-cell and after that absorbs the contents; meanwhile the plasm-body is strained and seems to become stiff, the long pseudopodia being considerably shortened and sometimes entirely drawn back.

LLOYD, whose observations are based as well on film-pictures as on the study of living material, equally states, that bundles of pseudopods stretch themselves along the filament and pull it forcibly, but, according to him, they are drawn back before the filament breaks to pieces. LLOYD also agrees with GOBI in the opinion that a "food-receptive vacuole" occurs, but he gives an other explanation of its origin than the observer mentioned. According to LLOYD the animal touches the cell-wall, through which this wall is chemically changed such that it becomes soft in that place because the cellulose turns into hydrocellulose. Through turgor-pression this soft part of the wall bulges out and forms a "blister" in the plasm-body ot the animal, which is the same as the "food-receptive vacuole" of GOBI and the "voûte" of PENARD. The pushing away of the cells from or out of the filament is ascribed directly to the turgor-change of the cell-contents, in consequence of which fact the blister bursts. That cells were thrown off according to turgor-decrease in *Spirogyra* and *Mesocarpus* was already known (COHN, BENECKE). LLOYD finishes his statements with the remark that his observations have convinced him, that the pseudopods play no active part in the effective ingestion of the food. GOBI saw 9, LLOYD 5 cells being emptied one after another by the same animal. As to the pseudopods, which, according to GOBI, are totally or partially drawn back during the feeding, LLOYD adds, that CIENKOWSKY maintains, that they remain unchanged — this is, however, not true, see above p. 101 —, but that he himself has never observed that they were present during the whole of the process, although some were probably used as "anchors".

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[1] The periodical in which these were published is not to be had in any Netherland public library.

STUMP (1935) discovered, that some species of the Thecamoeba-genera *Difflugia, Pontigulasia* and *Lesquereusia* feed in some cases in the same way as *Vampyrella lateritia*; according to him there exist the following differences however. It is true that the filament of the weed bent or broke sometimes during the activity of the specimens of the mentioned species; but this was never accompanied by a sudden shock as was the case with *Vampyrella*. The way in which the cell-wall is opened, is also different: the Thecamoeba do not open the wall by suction, but by pulling; in this way irregular rents and flaws appear instead of round holes with smooth edges. The proper feeding takes place through the cytoplasmatic stream and not by a sucking action as in *Vampyrella*.

We discovered the same process in an other species of the genus *Difflugia, D. rubescens*; see for our observations of this case our publication in the "Proceedings" Vol. 44, 1941.

Earlier observations of one of us on *Vampyrella lateritia* (1907 a) and *Hyalodiscus rubicundus* (1907 b) also procured some information about the feeding of this species. In the well-known way *Vampyrella* fed exclusively on the cell-contents of a *Spirogyra* spec.; the process of ingestion of the food pretty well agreed with CIENKOWSKY's description of it. The cell-filament remained intact; neither were the cells isolated, nor was the filament hurled aside with a shock when the cell opened. During the feeding the pinhead-pseudopods were either active or drawn in and in the latter case only the longer ones in action, or both kinds drawn in. The feeding-process always lasted shorter than 20 minutes, generally ± 15 minutes. The holes in the cell-wall were large, round or elliptical and smooth-edged. *Hyalodiscus* only devoured the contents of *Oedogonium*-cells; here the proper feeding was finished in about 2 minutes. A remarkable variant of the process, not yet observed until then and further for all we know, was the following. After an *Oedogonium*-cell had been emptied, the animal sometimes introduced a broad, finger-formed pseudopodium into the empty cell, which attached itself to one of the separating side-walls and devoured the contents of the neighbouring cell through the hole thus made. Sometimes the contents of the adjacent cell at the other side were taken in the same way.

SAMENVATTING.

1. De beschreven populatie van *Vampyrella lateritia* voedde zich uitsluitend met den celinhoud eener *Mougeotia*-soort.
2. De tijdens de voedselopname gewoonlijk uitgestrekte lange pseudopodiën oefenden daarbij waarschijnlijk op de cellen van den draad een trekking uit, die op het oogenblik van het opengaan der cel sterke dislocaties van den draad ten gevolge had.
3. Het eigenlijke ledigingsproces liep in enkele minuten af.
4. Na afloop daarvan stak het dier een vingervormig pseudopodium in de leege cel, a.h.w. om deze op eventueele resten van haar inhoud te onderzoeken en deze nog op te nemen.
5. Gewoonlijk werden eenige cellen achter elkaar, de grootere tot een maximum van 3, de kleinere tot een van 7, geledigd.

6. De openingen in den celwand waren rond of breed-elliptisch en hadden gladde randen.

7. Een ingrijpende verandering van het plasma der wiercel vóór het doorbreken van den wand kon niet worden geconstateerd.

8. Het eigenlijke openingsproces van den wand der wiercel is waarschijnlijk de resultante van drie factoren: 1. een chemische verandering van den wand door het dier op een vrij scherp begrensde plek; 2. een druk van binnen uit door verhoogden turgor als gevolg van een sterke vacuolisatie; 3. een trekkende (zuigende) werking, uitgeoefend door het dier. De onderlinge verhouding van de grootte dezer componenten kan van geval tot geval variëeren.

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[1]) In our previous paper in the "Proceedings" (1940) we mentioned in the record of literature a publication of S. MIHAÉLOFF, communicated in the "Bulletin de l'Institut d'Egypte", Vol. **18**, 1936. We took this from a quotation of G. ENTZ Jr. in the "Archives Neérlandaises de Zoologie", T. III, 1938, as we did not know then the article itself. Having read it since we feel obliged to declare that it is obviously one of the worst cases of plagiarism anywhere to be found. It contains: 1. a statement on artificial amoebas; 2. a description of *Protamoeba primordialis* KOROTNEFF; 3. observations on the artificial production of Thecamoeba-shells; 4. observations on the ingestion of food by *Vampyrella lateritia*; 5. diagnoses of two "new" species of Protozoa, a Rhizopod and a Flagellate. The author asserts that his statements are based on his own observations; we, however, affirm emphatically that all the so-called observations of MIHAÉLOFF are actually taken from E. PENARD and that the words in which these "observations" are communicated are copied with some slight modifications from PENARD's publication: "Les Protozoaires considérés sous le rapport de leur perfection organique" (Genève, 1922), this authors name not being mentioned at all. The height of impudence is the way in which the two new species are created. The diagnose of the Rhizopod, *Amphiterma* (must be: *Amphitrema*) *aegyptica* n. sp., is an almost verbatim copy of PENARD l.c., but here it refers to *Amphitrema lemanense*, a species discovered by PENARD in the Lake of Geneva. The origin of the diagnosis of the Flagellate, *Sphaerulla* (must be: *Sphaerula*) *nili* n. sp., is even more remarkable. It is also found almost verbally in PENARD's publication, but it is composed by MIHAÉLOFF of the diagnoses of two new species described by PENARD, sc. *Cryptomonas ovata* and *Sphaeroeca* sp.

How to qualify this bare-faced falsification of a scientific text, baffles us completely, but let the conviction that this case is an unparalleled one be our comfort.

Psychologie. — *Das Problem des Ursprungs der Sprache.* I. Von G. RÉVÉSZ. (Communicated by Prof. A. P. H. A. DE KLEYN.)

(Communicated at the meeting of December 27, 1941.)

Inhalt:

1. Einleitung.
2. Das Ursprungsproblem.
3. Die Ursprungstheorien.
   - A. Die Ausdruckstheorie.
   - B. Die Interjektionstheorie.
   - C. Die Nachahmungstheorie.
   - D. Die Gebärdentheorie.
   - E. Die tierpsychologische Theorie.
   - F. Die ontogenetische Theorie.
   - G. Die bewusstseinspsychologische Theorie.
   - H. Ethnologie und Pathologie im Dienste des Ursprungsproblems.
4. Das Problem der Ursprache.
5. Die Sprache der Urmenschen.
6. Die prinzipielle Unhaltbarkeit der Ursprungstheorien.
7. Die Kontakttheorie.
8. Mensch und Sprache.

1. *Einleitung.*

Bei Betrachtung der menschlichen und tierischen Welt fällt uns eine ganz besondere Erscheinung auf, die für das anthropologische Grundproblem von der grössten Bedeutung ist.

Während die jetzt auf der Erde lebenden Tiere Jahrtausende, vermutlich 20—30.000 Jahre lang, in ihren Verhaltungsweisen, Trieben, Affekten, Bedürfnissen, Leistungen, sozialen Formen, ihrer psychobiologischen Beschaffenheit *keine Veränderung* zeigen, hat das Menschengeschlecht während dieser Zeit eine bedeutende *Geschichte* gezeigt. Tiergemeinschaften traten immer wieder auf, ohne nennenswerten Einfluss auf die folgenden Generationen auszuüben, während in dieser Zeit die Menschheit durch verschiedene Etappen hindurch gegangen ist und dabei eine Entwicklung durchgemacht hat, die durch Ueberlieferungen, Erfahrungen und Leistungen kollektiver und individueller Art bestimmt ist.

Der Elefant im Urwald hat sich vor 10.000 Jahren genau so verhalten wie jetzt. Er hat seinen Rüssel gerade so zum Greifen, zum Tasten, zum Trinken benützt wie heute. Seine Stosszähne dienten in der Tertiärzeit genau so zum Abreissen der Baumrinde, zum Aufwühlen des Bodens, wie jetzt. Die Bienen haben ihren Nahrungserwerb in der vorgeschichtlichen Zeit eben so zwangs- und zweckmässig organisiert und ihre Feinde eben so grausam verfolgt wie gegenwärtig. Auch der junge Esel sprang vor Tausenden von Jahren genau so munter und komisch umher wie heute, und die Krokodile dürften zur Zeit des Leviathans nicht viel liebenswürdiger gewesen sein als nun. Demgegenüber sah der Mensch in der paläolithischen Zeit anders aus; er besass eine andere Kultur als der geschichtliche Mensch. Auch der Franzose wird in der Epoche der Völkerwanderung eine andere, primitivere geistige Konstitution gehabt haben als in der Periode der grossen kulturellen Entwicklung seines Landes.

Diese Invariabilität der Tiere erklärt sich daraus, dass sie im höchsten Masse an die Umwelt gebunden sind, sich der Natur vollkommen unterwerfen und den geophysischen

Verhältnissen zwangsmässig anpassen. Sie werden vollkommen von ihren Trieben beherrscht, mithin von einem konservativen Prinzip, welches das Bestehende schützt und jeder Aenderung widerstrebt. Das Tier verändert in seinen Lebensumständen und Verhaltungsweisen aus eigenem Antrieb nichts; nur Naturereignisse und Aenderung der Umwelt (z.B. Domestikation) können das Tier zwingen seine Reaktionsweisen und Gewohnheiten zu ändern. Selbst die durch Kreuzung und Veredlung gezüchteten Tiere behalten die Eigentümlichkeiten ihrer wilden Artgenossen zum grossen Teil bei. Der Mensch jedoch tritt der Natur entgegen, emanzipiert sich von den naturgegebenen Bedingungen, gewinnt neue Bedürfnisse, die er dank seiner Erfindungsgabe zu befriedigen versucht. Zur Befriedigung seiner neuen Bedürfnisse schuf der Mensch in der Gemeinschaft die Nahrungszubereitung durch Feuer, die Viehzucht und Agrikultur, das Handwerk, die Kleidung und den Schmuck, ferner die Kunst, Religion und Sitte, das Recht und die Formen gesellschaftlicher Organisation. Das Tier blieb naturgebunden, während sich der Mensch von dem Zwang der Natur befreit hat, geistige und moralische Freiheit erwarb und dadurch Entwicklungsmöglichkeiten, die ihn von dem Tier und von seinen mutmasslichen tierischen Vorfahren prinzipiell trennen. An Stelle des alles beherrschenden konservativen Prinzips tritt eine grosse Plastizität, an Stelle der Triebziele bewusste Zielsetzungen, an Stelle eindeutig determinierter Triebkräfte Verstand, Vernunft und Wille.

2. *Das Ursprungsproblem.*

Wenn wir die Frage stellen, worauf der Unterschied beruht, was Mensch und Tier *unabänderlich* in alle Ewigkeit trennt, so ist meine Antwort: die *Sprache.* Die Sprache als Mittel der gegenseitigen Verständigung mit ihren Begriffen, Formen und Gesetzen, mit ihrer symbolischen Natur, ihrer engen Beziehung zum Denken und ihrer sozialen Bedeutung ist das, was den Menschen zum Menschen macht und von dem Tier prinzipiell scheidet.

Die Menschwerdung setzt mit der Sprache ein: *Ohne Sprache kein Mensch, ohne Mensch keine Sprache* ¹). Durch diesen Satz erhält die Frage nach dem Ursprung der Sprache ein besonderes Interesse und die Forschung einen ganz bestimmten Ausgangspunkt.

Wir sind in unseren Gedanken und Vorstellungen von der Idee der kontinuierlichen Entwicklung so stark beeinflusst, dass wir unsere Kenntnis einer relativ ausgebildeten Kulturerscheinung wie der Sprache solange lückenhaft finden, bis wir nicht für ihren Ursprung und ihre früheren Entwicklungsstufen eine annehmbare und logisch unanfechtbare Erklärung gefunden haben.

Der denkende und sich auf sich besinnende Mensch will nicht bei der Konstatierung stehen bleiben, das schon der „erste" Mensch die „Sprache Gottes" verstand und dass er seine Wünsche und Gedanken durch die Sprache bzw. durch sprachgebundene Gebärden mitzuteilen imstande war. Er richtet seine Aufmerksamkeit auf die *Vorgeschichte,* auf die mutmasslichen *Vorstufen* der Sprache, also auf jene Aeusserungen der menschlichen Vorfahren, die der eigentlichen Sprache vorangegangen sind. Man versucht jene *vorsprachlichen* Stadien zu rekonstruieren, die ihren Endpunkt gerade dort haben, wo der Anfangspunkt der Sprache liegt.

Ueber diese Vorstadien wissen wir unmittelbar überhaupt nichts, und wir werden darüber auch niemals etwas Positives erfahren. Für die Erkenntnis des Ursprungs der

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¹) Aehnliche Gedanken führen W. VON HUMBOLDT in seiner Studie „Ueber das vergleichende Sprachstudium" (1820) und auch H. DELACROIX in „Le Langage et la Pensée" aus (Paris, 1930, S. 218 ff.). Auch ERNST RENAN vertritt diese Auffassung. Er drückt sich einmal folgendermassen aus: C'est donc un rêve d'imaginer un premier état où l'homme ne parle pas, suivi d'un autre état où il conquit l'usage de la parole. L'homme est naturellement parlant, comme il est naturellement pensant." (De l'origine du langage. Paris, 1859.)

Sprache fehlt uns jede sprachgeschichtliche Grundlage. Der Zeitabschnitt, in welchem die Vorgeschichte der Sprache anzusetzen sein würde, liegt mehrere hunderttausend Jahre hinter uns. Dieser Umstand schliesst nicht aus, dass man über den mutmasslichen vorsprachlichen Zustand des neo- oder palaeolithischen „Menschen" Hypothesen aufstellt und gar prüft, ob nicht vielleicht bei gewissen hochorganisierten Tieren bereits Hinweise auf den Ursprung der Sprache zu finden sind. Die Fruchtbarkeit der Entwicklungsidee soll sich bei diese Frage erweisen.

Indem wir so grossen Wert auf die Vorgeschichte legen, wollen wir zum Ausdruck bringen, dass es sich beim Ursprungsproblem nicht um die Urformen, um die elementaren Formen der *Sprache* handelt, sondern um die Erkenntnis des *Urgrundes*, aus dem die Sprache gleichsam geboren ist, also um jene Aeusserungen, die für die *Entstehung* der Sprache massgebend gewesen sein, die ersten sprachlichen Ausdrucksweisen veranlasst haben sollen. Hinsichtlich der phylogenetischen Entwicklung der Sprache müssen wir also zwischen der *Entstehung* oder des *Ursprungs* und der *Fortentwicklung* der Sprache unterscheiden. Das Problem der Entstehung bezieht sich auf die hypothetische Vorgeschichte der Sprache, auf die Rekonstruktion der vorsprachlichen Formen, aus denen die Sprache hervorgegangen ist, während es sich bei dem Problem der Fortentwicklung um jene Sprachformen handelt, durch welche die Sprache von ihren Anfängen, von ihren primitivsten Manifestationen bis zur Vollsprache hindurchgehen musste.

Unser Wissensdrang hat inbetreff des Ursprungs der Sprache zu verschiedenen Annahmen geführt. Man suchte die Sprache herzuleiten von Ausdrucksbewegungen und emotionellen Lautäusserungen, von der Imitation von Naturlauten von den Tierlauten, der Kindersprache und den Sprachen primitiver Völkerstämme.

Schon die aussergewöhnliche Buntheit der bei den Lösungsversuchen angewandten Gesichspunkte weist darauf, dass man das Problem nicht scharf ins Auge gefasst hat. Man vergegenwärtigte sich nicht, was man eigentlich erforschen wollte. Manche Hypothesen beziehen sich nämlich auf die *Vorgeschichte* der Sprache, wie die Ausdrucks- und Nachahmungstheorie, sowie die Theorie, welche die Sprache aus Tierlauten abzuleiten strebt, während andere Hypothesen, z.B. die, welche die Sprache von Gebärden herzuleiten trachtet, ferner die ethnologische und kinderpsychologische Hypothese sich auf die *Entwicklungsstufen* der Sprache richten.

Beruhte die Schwierigkeit nur auf der Zweideutigkeit der Problemstellung, dann würde unsere Aufgabe eine sehr leichte sein: man müsste einfach die Entwicklungshypothesen aus der Problematik ausschalten und bloss die Ursprungstheorien einer Kritik unterwerfen. Der Grundfehler liegt jedoch darin, dass man sich bei der Theorienbildung keine Rechenschaft davon abgelegt hat, welche Forderungen man an die hypothetischen Vorstufen im allgemeinen stellen muss. Wenn man die Hypothesen überblickt, so gewinnt man den Eindruck, dass hier mehr die wissenschaftliche Eingebung als eine auf Prinzipien und Methoden eingehende Überlegung das Entscheidende war. Allerdings ist es von vornherein nicht auszumachen, auf welchem Wege man zum Resultat gelangt; man darf auch den Gewinn nicht geringschätzen, der aus solchen methodologischen Bestrebungen für die Ursprungsforschung selbst im Falle eines negativen Resultats erwächst. Andererseits müssen wir uns deutlich machen, welche *Art* von entwicklungsgeschichtlichen Tatsachen als Vorstufen der Sprache in Betracht kommen können und welche prinzipiell auszuschliessen sind.

Will man demnach die vorgeschichtlichen Stufen einer Funktion oder einer Fähigkeit feststellen bezw. rekonstruieren, gleichsam aus anderen, früheren, ableiten — wie man in unserem Falle die Sprachfunktion auf eine andere und allgemeinere zurückzuführen versucht, — und legt man dabei auf eine wissenschaftlich berechtigte Hypothesenbildung Wert, so müssen gewisse theoretische Forderungen gestellt werden, von deren Erfüllung der Ertrag der Forschung abhängt.

Bei der Ursprungsforschung im allgemeinen, folglich auch bei der Frage nach der Entstehung der Sprache, muss man zunächst danach streben, jenes Stadium festzulegen, das *unmittelbar* der Entstehung der betreffenden Erscheinung, hier also der Sprache, voranging und von dem man voraussetzt, dass es bei ihrem Zustandekommen eine entscheidende Rolle gespielt hat. Unsere Aufgabe ist es mithin, nach jenen Ausdrucksformen zu suchen, die zwar selbst noch keine sprachlichen Gebilde darstellen, jedoch bei der Entstehung der Sprache massgebend gewesen sind. Die Festlegung dieses unmittelbar *vorangehenden* Stadiums ist von grosser Bedeutung; denn je weiter man von dieser *Vorstufe* abrückt, desto hypothetischer und willkürlicher werden die supponierten Vorstufen. Dies gilt nicht nur bezüglich der Sprache, sondern bezüglich aller menschlichen Funktionen und Fähigkeiten. Beleuchten wir dies an einem Beispiel.

Man setze den Fall, dass es uns gelingt, die unmittelbare Vorstufe des künstlerichen Schaffens der Menschheit zu bestimmen. Unsere Wissbegierde würde uns veranlassen, *vor* dieser Stufe liegenden Perioden der geistigen Entwicklung zu erforschen, um die elementaren Kräfte der Formgestaltung im allgemeinen zu erschliessen. Zu diesem Zwecke müssten wir tief in die Vorgeschichte der Menschheit herabsteigen. Wir werden in den uralten Zeiten solche menschliche Erzeugnisse finden, die man wegen ihrer äusserlichen Ähnlichkeit oder wegen ihrer Verwendung allzu leicht zu Kunstobjekten späterer Kulturepochen in entwicklungsgeschichtliche Beziehung bringt. Die primitivsten Linienverzierungen an Gefässen, die Formvariationen von Waffengeräten genügen, um in diesen Artefacta deutliche Manifestationen der Kunst zu erblicken. Die Entstehung der Kunst wird demgemäss weit in die mittelpaläolithische Zeit zurückverlegt, in die erste Periode der Menschwerdung. Ob eine künstlerische Absicht, ein Kunstwollen bei der Verfertigung jener Objekte vorlag, ob damals die Absicht bestand, aesthetisch Wertvolles zu schaffen, danach wird nicht gefragt. In dieser Weise werden Scheiben mit konzentrischen Linien, allerprimitivste Darstellungen von menschlichen Figuren, schreckenerregende Götterbilder und böse Geister, Amulette, magische Objekte, selbst einfachste Hausgeräte ohne weiteres als Uranfänge der Kunst betrachtet. Diese kritiklose, ausschliesslich von Äusserlichkeiten bestimmte Tendenz, artungleichen Leistungen verwandte Züge zuzusprechen, hat ihren Grund darin, dass man hierbei die *Formgebung* schlechthin mit der *künstlerischen* Formgebung identifiziert. Man lässt dabei gänzlich ausser Acht, dass diese pseudo-künstlerischen Erzeugnisse das natürliche Produkt der autonom formenden Menschenhand sind. Diese verleiht mit geradezu biomechanischer Notwendigkeit den verfertigten Objekten Formen, die in den einfachsten Erzeugnissen ebenso in Erscheinung treten, wie in den höchsten Kunstleistungen. Zahlreiche Beispiele liefert die Keramik, dieses älteste Handwerk der Menschheit.

Bei dem Studium der Entstehungsgeschichte der Sprache kommen wir in dieselbe Lage, wenn wir nicht von jenen Tätigkeiten ausgehen, von denen wir mit grosser Wahrscheinlichkeit annehmen dürfen, das sie der Sprache *unmittelbar* vorangegangen sind.

Es entsteht nun die wichtige Frage, welche Eigenschaften, Merkmale eine Ausdrucksform besitzen muss, um sie als *unmittelbaren* Vorläufer einer differenzierteren Funktion oder Tätigkeit gelten lassen zu können. Unserer Ansicht nach muss sie zwei *Prinzipien* gehorchen: einmal dem *Prinzip der gemeinsamen konstitutiven Merkmale*, sodann dem der *einheitlichen Tendenz*.

Die Anwendung des ersten Prinzips bedeutet, dass man von einer Vorstufe erst dann sprechen darf, wenn Merkmale vorhanden sind, die auch für die differenziertere Funktion von konstitutiver Bedeutung sind. Diese Merkmale können sich auf phänomenale Aehnlichkeit, gelegentlich auch auf strukturelle Uebereinstimmung beziehen.

Im allgemeinen wird für die Rekonstruktion der Vorstufen der Nachweis der Aehnlichkeitsbeziehung genügen. Diese kann nämlich so evident sein, dass die genetische Bedeutung der weniger differenzierten Funktion, also die der früheren Stufe, ausserhalb allen Zweifels steht. Andererseits ist es uns bekannt, dass gelegentlich die konstitutive Bedeutung der Einzelmerkmale für das Zustandekommen der höheren Funktion schwer zu beurteilen ist. So wird man z.B. auf Schwierigkeiten stossen, will man entscheiden, ob die Fähigkeit, Laute nachzuahmen, als Vorstufe des Sprechens anzusehen ist; denn ob zwischen den Nachahmungslauten und der menschlichen Sprache eine innere Verwandtschaft besteht, lässt sich trotz der auffallenden Uebereinstimmung, die zwischen beiden

Aeusserungen ohne Zweifel vorliegen, nicht beurteilen. Diese Frage kann erst dann beantwortet werden, wenn wir unser *zweites Prinzip* zu Hilfe nehmen und an Hand von ihm untersuchen, ob beide Tätigkeiten durch eine gemeinsame Grundeinstellung oder durch eine gemeinsame Grundtendenz miteinander verbunden sind oder nicht. Von diesem Gesichtspunkt aus beurteilt werden wir zwischen Nachahmungs- oder Naturlauten und Worten keine innere Beziehung annehmen können.

Die Grundtendenz vermag sich in verschiedener Weise kundzugeben, z.B. in einem spezifischen Streben, in einem Bedürfnis allgemeiner oder in einer Funktionsweise besonderer Art. Lässt sich eine solche Tendenz auffinden und lassen sich dabei noch übereinstimmende Merkmale aufzeigen, so sind wir berechtigt, zwischen beiden Funktionen eine fortschreitende Entwicklung anzunehmen und die differenziertere, spätere, aus der weniger differenzierten, früheren, abzuleiten.

Es ist klar geworden, dass die entwicklungsgeschichtliche Beziehung zwischen den mutmasslichen Vorstufen und der daraus abgeleiteten ausgereiften Funktion auf Uebereinstimmung beruhen muss, die sich auf gemeinschaftliche essentielle Merkmale gründet und von einem gemeinsamen Aufbau- oder Funktionsprinzip unterstützt bezw. bestimmt wird. Dass gelegentlich auch das eine oder das andere genügt die genetische Beziehung festzustellen, geht aus den Vorangegangenen hervor. Eine Regel dafür, wann nur ein und wann beide Prinzipien erforderlich sind, lässt sich nicht aufstellen. Das hängt von dem Gebiet und von der Art der Vorstufen ab, ferner davon, ob sich Uebergänge zwischen der Vorstufe und der ausgereiften Funktion finden lassen. Vermag man die beiden Aeusserungsformen nicht unter einem einheitlichen Prinzip zu bringen, so verringert sich jedenfalls die Ueberzeugungskraft der Ableitung.

Diese methodologische Regel gilt nicht nur für die psychischen Funktionen, sondern ebenso für die Entstehung der verschiedensten Erscheinungen und Werte in der sozialen und kulturellen Entwicklung. Z.B. die Musik kann unmittelbar nur aus Ausdrucksformen entstanden sein, die konstituierende Merkmale der Musik enthalten, und ebenso müssen sich die menschliche Gesellschaftsformen aus Verbänden herausentwickelt haben, die solche sozialen Elemente oder Kräfte in sich trugen, die bei allen menschlichen Gesellschaftsformen konstitutiv wirksam sind. So fügt sich auch ein neuer Stil nur dann in die Kunstentwicklung organisch ein, wenn er eine innere Beziehung zu den früheren Stilarten hat, d.h. mit ihnen durch gemeinsame konstitutive Merkmale oder durch gemeinsame künstlerische Prinzipien oder durch beide verbunden ist. Dies schliesst die Wirksamkeit von neuen Gesichtspunkten und neuen Zielsetzungen natürlich keineswegs aus.

Aus diesen Ueberlegungen ergibt sich, dass wir erst dann berechtigt sind, gewisse Tätigkeiten oder Funktionen als Vorläufer der Sprache anzusehen, wenn es sich unzweifelhaft zeigen lässt, dass sie den genannten Forderungen entsprechen.

Als essentielle Merkmale der Sprache kommen in erster Reihe die *Grundfunktionen* in Betracht, also die Mitteilung, Kundgebung und Darstellung. Von der Bezeichnungsfunktion kann man hier absehen, da sie nicht zu den allgemein notwendigen Charakterzügen der Spracharten gehört. Die Gebärdensprache z.B. entbehrt diese Funktion. Ausser den Grundfunktionen können noch die artikulatorische und grammatische Struktur der Sprache, ferner das phonetische System, bzw. die sinnlich-anschaulichen Ausdrucksgebärden als Zeichen der inneren Verwandtschaft zwischen Sprache und Vorstufe betrachtet werden.

Was das allgemeine Prinzip betrifft, so weisen wir vorwegnehmend auf das Bedürfnis eines gegenseitigen *Kontaktes* hin, das als Grundbedingung aller Kommunikationsformen, folglich als Bindeglied zwischen der Sprache und ihren Vorstufen, zu gelten hat.

Durch das Aufzeigen der gegenseitigen Beziehung ist die Frage nach der zeitlichen Aufeinanderfolge noch nicht entschieden und ebenso ist es nicht gesagt, dass das Abgeleitete eine höhere Entwicklungsstufe des Ursprünglicheren ist. Die Entwicklung kann sprunghaft gewesen sein; auch vermag die spätere Form eine Regression, eine Rückbildung, darzustellen.

Bei der Rekonstruktion der Entwicklungsgeschichte einer fundamentalen Funktion des menschlichen Geistes muss man sehr vorsichtig vorgehen und sich immer vergegenwärtigen, dass man dabei lediglich auf Rückschlüsse und Analogien angewiesen ist. Man darf sich weder *durch* Aehnlichkeiten imponieren lassen noch *ohne* nachweisbare Aehnlichkeit einen entwickelteren Zustand aus einem primitiven ableiten, ohne zuvor genau erwogen zu haben, ob die oben erwähnten prinzipiellen Voraussetzungen erfüllt sind.

Ganz besonders muss hierauf geachtet werden, sobald es sich um solche entwicklungsgeschichtliche Probleme handelt, bei denen man weit über die Grenzen der empirischen Kenntnisse und Ueberlieferungen hinausgreifen muss. Eine kritische Einstellung und ein Verständnis für die geschichtliche Entwicklung sind da besonders notwendig, um das Gleichgewicht zwischen Erfahrung und Konstruktion herzustellen. Dies alles gilt im besonderen für die Sprache. Obgleich auf der einen Seite sich die Bedingungen der *Fortentwicklung* vielleicht bei keinem Kulturgut mit solcher Genauigkeit erkennen lassen wie bei der Sprache (H. PAUL), gibt es auf der anderen Seite vielleicht kein Kulturgut, hinsichtlich dessen *Entstehung* der geistes- und kulturwissenschaftlichen Forschung so wenig Tatsachen zur Verfügung stehen, wie es gerade bei der Sprache der Fall ist. Denkmäler und Erzeugnisse der prähistorischen Zeit, die über das geschichtliche Werden der Menschheit in fernliegenden Zeiten Aufklärung geben, liefern über die Sprache keine Aufschlüsse. Wir sind folglich darauf angewiesen, aus *manifesten Aeusserungen des rezenten Menschen die vorsprachliche Periode zu rekonstruieren.* Dabei gehen wir von der Voraussetzung aus, dass der gegenwärtige Mensch in seinem sozialen Umgang noch gewisse archaische Kontaktformen verwendet, die er aus seiner „sprachlosen Zeit" in die jetzige hinübergerettet hat. Wir nehmen ferner an, dass diese Aeusserungsformen die Entstehungsgrundlage der menschlichen Sprache bilden. Wie diese primäre Aeusserungen entwicklungsgeschichtlich verwertet werden können und ob sie sich auch noch weiter nach der animalischen Periode hin zurückverfolgen lassen, soll sich später zeigen.

Zunächst wollen wir die mannigfachen Ursprungs- bzw. Entwicklungstheorien einer kritischen Behandlung unterwerfen. Unsere Auffassung von dem Ursprung der Sprache wird hierbei deutlich hervortreten. Im Anschluss daran werden wir Gelegenheit haben unsere Ursprungstheorie zu entwickeln.